CHAPTER

4

APPLYING THE EXPONENTIAL AND TRIGONOMETRIC FUNCTIONS

Syllabus coverage

TOPIC 3.1: FURTHER DIFFERENTIATION AND APPLICATIONS

Exponential functions

3.1.1 estimate the limit of $\frac{a^h - 1}{h}$ as $h \to 0$, using technology, for various values of $a > 0$

3.1.2 identify that e is the unique number a for which the above limit is 1

3.1.3 establish and use the formula $\frac{d}{dx}(e^x) = e^x$

3.1.4 use exponential functions of the form Ae^{kx} and their derivatives to solve practical problems

Trigonometric functions

3.1.5 establish the formulas $\frac{d}{dx}(\sin x) = \cos x$ and $\frac{d}{dx}(\cos x) = -\sin x$ by graphical treatment, numerical estimations of the limits, and informal proofs based on geometric constructions

3.1.6 use trigonometric functions and their derivatives to solve practical problems

Differentiation rules

3.1.9 apply the product, quotient and chain rule to differentiate functions such as xe^x, $\tan x$, $\frac{1}{x^n}$, $x\sin x$, $e^{-x}\sin x$ and $f(ax - b)$

TOPIC 3.2: INTEGRALS

Anti-differentiation

3.2.4 establish and use the formula $\int e^x\,dx = e^x + c$

3.2.5 establish and use the formulas $\int \sin x\,dx = -\cos x + c$ and $\int \cos x\,dx = \sin x + c$

Applications of integration

3.2.18 calculate total change by integrating instantaneous or marginal rate of change

3.2.19 calculate the area under a curve

3.2.20 calculate the area between curves determined by functions of the form $y = f(x)$

3.2.21 determine displacement given velocity in linear motion problems

3.2.22 determine positions given linear acceleration and initial values of position and velocity

Video playlists (6):

- **4.1** Exponential growth and decay
- **4.2** Differentiating exponential functions
- **4.3** Integrating exponential functions
- **4.4** Differentiating trigonometric functions
- **4.5** Integrals of trigonometric functions
- **WACE question analysis** Applying the exponential and trigonometric functions

Worksheets (7):

- **4.1** Exponential functions
- **4.2** Derivatives of exponential functions
- **4.4** Further optimisation problems • Trigonometric functions and gradient
- **4.5** Finding indefinite integrals 1 • Finding indefinite integrals 2 • Finding definite integrals

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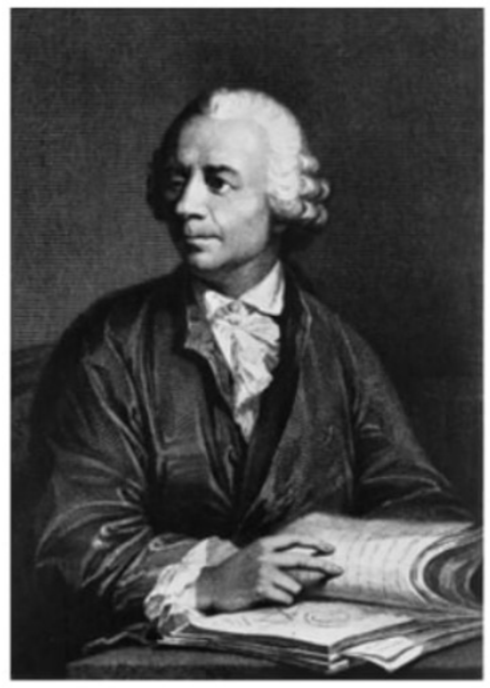
4.1 Exponential growth and decay

Euler's number, *e*



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The function $y = a^x$, where the base a is a positive constant but not equal to 1, is called an **exponential function**. There is a special value of a, called e, discovered by the Swiss mathematician Leonhard Euler in 1731.

$e = 2.718\,28\ldots$

Euler's number is defined as $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$.

Euler is pronounced 'oiler'.

This number is used in compound interest calculations, where n represents the number of times the interest is compounded in a year. The table shows that as n becomes larger and approaches infinity, and the compounding of interest becomes continuous, the value of $\left(1+\frac{1}{n}\right)^n$ approaches 2.718 28…

Video playlist Exponential growth and decay

Worksheet Exponential functions

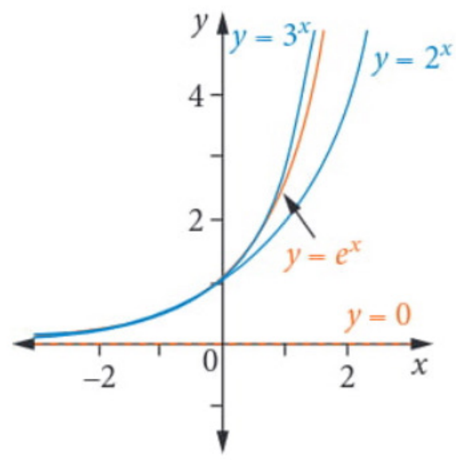
Frequency of interest compounding	n	$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$
Yearly	1	$\left(1+\frac{1}{1}\right)^1 = 2$
Half-yearly	2	$\left(1+\frac{1}{2}\right)^2 = 2.25$
Quarterly	4	$\left(1+\frac{1}{4}\right)^4 = 2.441\,406\,25$
Monthly	12	$\left(1+\frac{1}{12}\right)^{12} \approx 2.613\,035\,2902\ldots$
Weekly	52	$\left(1+\frac{1}{52}\right)^{52} \approx 2.692\,596\,954\,44\ldots$
Daily	365	$\left(1+\frac{1}{365}\right)^{365} \approx 2.714\,567\,482\,02\ldots$
Hourly	8760	$\left(1+\frac{1}{8760}\right)^{8760} \approx 2.718\,126\,690\,63\ldots$
Every minute	525 600	$\left(1+\frac{1}{525\,600}\right)^{525\,600} \approx 2.718\,279\,215\ldots$
Every second	31 536 000	$\left(1+\frac{1}{31\,536\,000}\right)^{31\,536\,000} \approx 2.718\,282\,472\,54\ldots$

The table shows that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$ and it can also be shown as $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$.

The exponential function

The function $y = e^x$ is called the **natural exponential function**. Things that grow naturally, such as trees or population size, follow the natural exponential function. e is an irrational number like π.

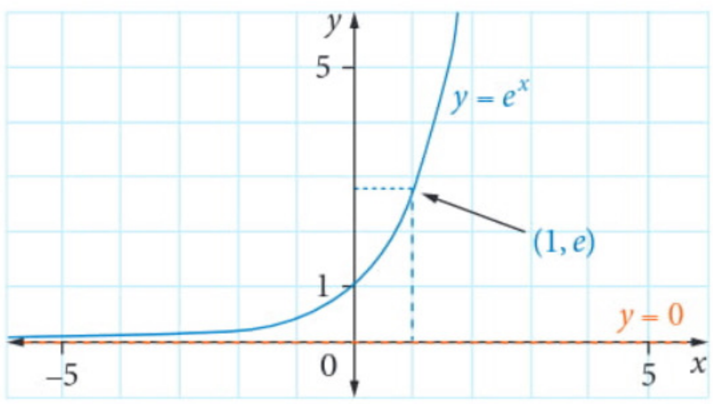
Since $2 < e < 3$, the graph of the natural exponential function lies between the graphs of $y = 2^x$ and $y = 3^x$, as shown.



The value of e is stored on the calculator.

This table of values for $y = e^x$ shows y values rounded to three decimal places, and its graph is shown below.

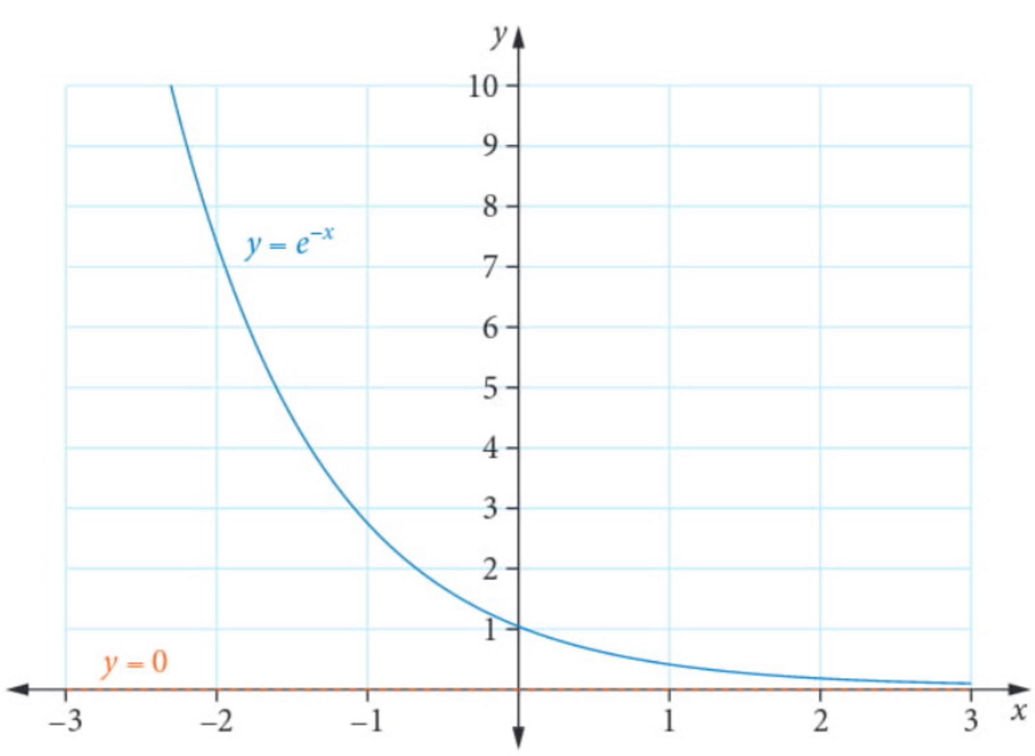
x	−3	−2	−1	0	1	2	3
y	$e^{-3} \approx 0.050$	$e^{-2} \approx 0.135$	$e^{-1} \approx 0.368$	$e^0 = 1$	$e \approx 2.718$	$e^2 \approx 7.389$	$e^3 \approx 20.086$



Properties of the natural exponential function, $y = e^x$

- It is a strictly increasing function, increasing slowly at first, then more quickly.
- The gradient of the graph is always increasing.
- The y-intercept is 1 (because $e^0 = 1$).
- The x-axis ($y = 0$) is a horizontal asymptote.

The graph of $y = e^{-x}$ is shown below.



Exponential growth and decay

Any quantity that **increases** according to the exponential function $y = A(e^{kx})$, where $k > 0$, is showing **exponential growth**. Something that grows exponentially increases slowly at first, then more quickly. Examples of exponential growth are population size (people, animals, bacteria), an investment attracting compound interest, a flu or computer virus, and the size of a bushfire.

Any quantity that **decreases** according to the exponential function $y = A(e^{kx})$, where $k < 0$, is showing **exponential decay**. Something that decays exponentially decreases quickly at first, then more slowly. Examples of exponential decay are radioactive decay, the cooling of substances, the intensity of light in water, and the dampening of vibrations.

4.1

Exponential growth and decay

The general form of an exponential growth or decay function is $N(t) = N_0e^{kt}$,
where N_0 is the initial value and the value of k determines the rate of growth or decay.
For a growth function, $k > 0$.
For a decay function, $k < 0$.

WORKED EXAMPLE 1 Modelling an exponential growth problem

The number N of gum trees with pink flowers in a region of Western Australia was studied.
The equation $N(t) = 1200e^{0.07t}$ was given as a model for the number of gums, with time t being the number of years since the study began.

a How many trees were there at the beginning of the study?

b How many trees were there after 10 years?

Steps	Working
a **1** This is an example of exponential growth. Substitute $t = 0$ into $N(t) = 1200e^{0.07t}$.	$N(t) = 1200e^{0.07\times0}$ $= 1200e^0$ $= 1200$ N_0 is always the initial value of $N(t) = N_0e^{kt}$.
2 Answer the question.	There were 1200 trees at the beginning.
b **1** Substitute $t = 10$ into $N(t) = 1200e^{0.07t}$.	$N(10) = 1200e^{0.07\times10}$ $= 1200e^{0.7}$ $= 2416.503\ldots$
2 Answer the question.	There were about 2417 trees after 10 years.

WORKED EXAMPLE 2 Finding the parameters in an exponential decay problem

A patient is given 250 mg of an anti-inflammatory drug. Each hour, the amount of drug in the person's system decreases exponentially so that after 1 hour there is 200 mg in the patient's system. The number of mg of the drug D in the patient's system after t hours is given by the rule $D(t) = D_0e^{-kt}$.

a Find the value of D_0.

b Find the value of k correct to four decimal places.

c Find the number of mg of the drug in the patient's system, correct to one decimal place, after 4 hours.

Steps	Working
a Substitute $t = 0$ into $D(t) = 250$.	$D(0) = 250$ $D(t) = D_0e^{-kt}$ $250 = D_0e^0$ $D_0 = 250$
b Substitute $t = 1$ into $D(t) = 200$ and solve using CAS.	$D(t) = 250e^{-kt}$ $200 = 250e^{-k}$ $k = 0.2231\ldots$
c **1** Substitute $t = 4$ into $D(t) = 250e^{-0.2231t}$.	$D(t) = 250e^{-0.2231t}$ $D(t) = 250e^{-0.2231\times4} = 102.4\ldots$
2 Answer the question.	There is 102.4 mg of the drug in the patient's system after 4 hours.

WORKED EXAMPLE 3 Using simultaneous equations to find the parameters in an exponential growth function

The diameter, in centimetres, of a species of gum tree $d(t)$ after t years is given by the rule $d(t) = d_0e^{mt}$. The diameter is 40 cm after 1 year, and 80 cm after 3 years.

a Write two equations that can be used to find the constants d_0 and m.

b Calculate the values of the constants d_0 and m, correct to three decimal places.

Steps	Working
a Substitute $t = 1$ into $d(t) = 40$ and $t = 3$ into $d(t) = 80$ where $d(t) = d_0e^{mt}$.	$d(1) = 40$ $d(t) = d_0e^{mt}$ $40 = d_0e^{m}$ equation 1 $d(3) = 80$ $80 = d_0e^{3m}$ equation 2
b Solve the simultaneous equations using CAS.	$d_0 = 28.283$, $m = 0.347$

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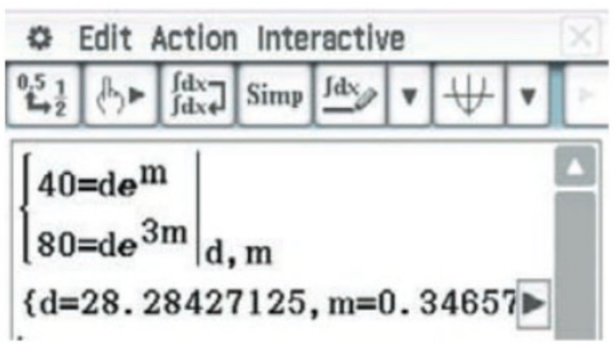

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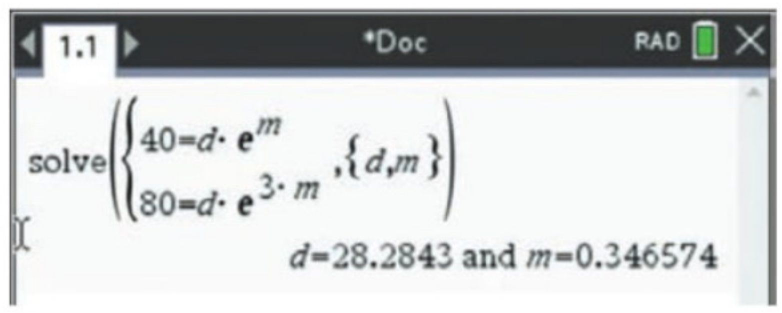


EXERCISE 4.1 Exponential growth and decay

ANSWERS p. 394

Mastery

1 WORKED EXAMPLE 1 The number of people, N, who have the flu virus at time t months is given by $N(t) = N_0e^{kt}$, where t is the number of months after the outbreak of the virus.

If the number is initially 200 and the number increases to 500 after 1 month, find

a **i** the value of N_0

ii the value of k, correct to four decimal places

b the number of people infected after 6 months.

2 A biological culture, in a laboratory, contains 100 000 bacteria at 1 pm on Monday. The culture grows exponentially so that the number of bacteria B after t hours after 1 pm Monday is given by the function $B(t) = B_0e^{kt}$. The culture has 105 000 bacteria at 6 pm on Monday. Find

a **i** the value of B_0

ii the value of k, correct to four decimal places

b the number of bacteria at 1 pm the following Tuesday

c the number of hours, to the nearest hour, for the number of bacteria to double.

3 WORKED EXAMPLE 2 An adult takes 400 mg of ibuprofen. After 1 hour, the amount of ibuprofen in the person's system is 280 mg. The number of mg of the drug D in the patient's system after t hours is given by the rule $D(t) = D_0e^{-kt}$.

a Find the value of D_0.

b Find the value of k correct to four decimal places.

c Find the number of mg of the drug in the patient's system, correct to one decimal place, after 2 hours.

4 The population of Doreen can be modelled by $P(t) = 6191e^{0.04t}$, where t is the number of years since 1990.

a What was the population in 1990?

b What was the population in 1991?

c By what percentage did the population increase in the first year?

5 WORKED EXAMPLE 3 The diameter d, in centimetres, of a species of elm tree after t years is given by the rule $d(t) = d_0e^{mt}$. The diameter is 10 cm after 1 year, and 15 cm after 2 years.

a Write two equations that can be used to find the constants d_0 and m.

b Calculate the values of the constants d_0 and m, correct to three decimal places.

6 In 1985, there were 285 mobile phone subscribers in the small town of Centerville and by 1987 this had grown to 873. The number of subscribers S, t years after 1984 is found to grow exponentially and is given by the rule $S(t) = S_0e^{kt}$.

a Find the value of S_0 to the nearest integer.

b Find the value of k, correct to four decimal places.

c Find the number of mobile phone subscribers in Centerville in 1995.

Calculator-free

7 (3 marks) Cobalt-60 is a radioactive substance whose decay rate can be modelled by the formula $P = P_0e^{kt}$, where P is the mass in grams, t is measured in days, P is the original amount and k is a constant. The time taken to decay to half of the original amount is known as half-life.

The half-life of Cobalt-60 is 5 years and its initial mass is 200 grams.

a Find the value of P_0. (1 mark)

b Show that $e^{-5k} = 2$. (2 marks)

Calculator-assumed

8 © SCSA MM2016 Q9ab (5 marks) Fermium-257 is a radioactive substance whose decay rate can be modelled by the formula $P = P_0e^{kt}$, where P is the mass in grams and t is measured in days and P_0 = original amount and k is a constant. The time taken to decay to half of the original amount is known as half-life. The half-life of Fermium-257 is 100.5 days.

a Determine the value of k to three significant figures. (3 marks)

b How many days will it take for 100 grams of the substance to first decay below five grams? (2 marks)

9 © SCSA MM2018 Q9ab (3 marks) The concentration, C, of a drug in the blood of a patient t hours after the initial dose can be modelled by the equation below.

$$C = 4e^{-0.05t}\ \text{mg/L}$$

Patients requiring this drug are said to be in crisis if the concentration of the drug in their blood falls below 2.5 mg/L.

A patient is given a dose of the drug at 9 am.

a What was the concentration in the patient's blood immediately following the initial dose? (1 mark)

b What is the concentration of the drug in the patient's blood at 11:30 am? (2 marks)

10 (5 marks) A radioactive element has a half-life of 10 minutes, meaning it takes 10 minutes for half the remaining atoms to decay. Originally, there are 8.0×10^{20} atoms in a sample of the element. The decay is modelled by $N = N_0 e^{-kt}$, where N_0 is the original number of atoms, N is the number of atoms present at time t, and k is the decay rate of the material.

a Find the value of k, correct to four decimal places. (2 marks)

b Find the number of atoms left after 1 hour. (2 marks)

c Is it possible to calculate when there is no sample left? (1 mark)

4.2 Differentiating exponential functions

Video playlist Differentiating exponential functions

Worksheet Derivatives of exponential functions

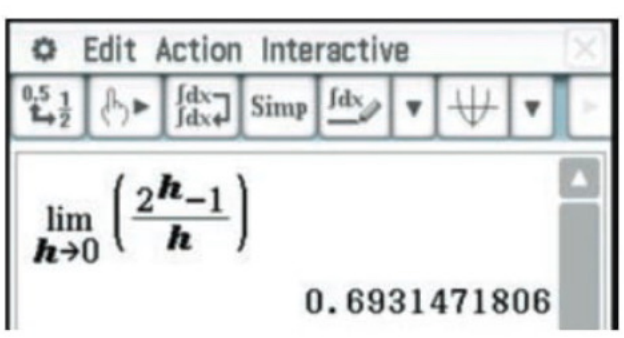
The derivative of $f(x) = 2^x$ by first principles is shown below.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{2^{x+h} - 2^x}{h} \\ &= \lim_{h\to 0} \frac{2^x \times 2^h - 2^x}{h} \\ &= \lim_{h\to 0} \frac{2^x(2^h - 1)}{h} \\ &= 2^x \left(\lim_{h\to 0} \frac{2^h - 1}{h} \right) \end{aligned}$$

We can factorise the 2^x because it does not involve h.

We can use CAS to evaluate the limit $\lim\limits_{h\to 0} \dfrac{(2^h - 1)}{h}$. Set your calculator to give decimal answers. (Decimal mode for ClassPad settings and Approximate calculation mode for the TI-Nspire.)

ClassPad



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This means that the derivative of $f(x) = 2^x$ is $2^x(0.6931\ldots) = (0.6931\ldots) \times 2^x$.

So, the derivative of $f(x) = 2^x$ is 2^x multiplied by a constant.

Similarly, the derivative of $f(x) = 3^x$ is $3^x \lim_{h\to 0} \frac{(3^h - 1)}{h}$.

When $\lim_{h\to 0} \frac{3^h - 1}{h}$ is evaluated, the answer is 1.0986…

This means that the derivative of $y = 3^x$ is $(1.0986\ldots) \times 3^x$.

Generally, the derivative of $y = a^x$ is a^x multiplied by a constant.

$$\frac{d}{dx}(2^x) = (0.6931\ldots)2^x$$

$$\frac{d}{dx}(3^x) = (1.0986\ldots)3^x$$

It would be convenient if this constant was equal to 1. Then the derivative of $y = a^x$ would be $\frac{dy}{dx} = a^x$ exactly.

WORKED EXAMPLE 4 Finding the value of a for which $\lim_{h\to 0} \frac{a^h - 1}{h} = 1$

Using CAS, find $\lim_{h\to 0} \frac{a^h - 1}{h}$ for a = 2, 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 2.7, 2.8, 2.9, 3.0.

Write your answers correct to four decimal places and, hence, determine the best approximation for a for which $\lim_{h\to 0} \frac{a^h - 1}{h} = 1$.

Steps	Working
1 Use CAS to find $\lim_{h\to 0} \frac{2^h - 1}{h}$.	$\lim_{h\to 0} \frac{(2^h - 1)}{h} = 0.6931\ldots$
2 Calculate the limit using $a = 2.1$. $\lim_{h\to 0} \frac{2.1^h - 1}{h}$ Repeat this process for the other values of a to complete the table.	see table below
3 Find the closest value of a for which $\lim_{h\to 0} \frac{a^h - 1}{h} = 1$.	The value of a for which $\lim_{h\to 0} \frac{a^h - 1}{h} = 1$ is $a = 2.7$.

a	$\lim_{h\to 0} \frac{a^h - 1}{h}$
2	0.6931…
2.1	0.7419…
2.2	0.7885…
2.3	0.8329…
2.4	0.8755…
2.5	0.9163…
2.6	0.9555…
2.7	0.9933…
2.8	1.0296…
2.9	1.0647…
3	1.0986…

From the above, it looks like the base a must lie somewhere between 2.7 and 2.8 and it turns out that this value is $e = 2.718\,28\ldots$, Euler's number.

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1$$

The derivative and integral of e^x

The natural exponential function is the function $y = e^x$.

The derivative of e^x is e^x: $\frac{d}{dx}(e^x) = e^x$

The derivative of e^{ax-b} is ae^{ax-b}: $\frac{d}{dx}(e^{ax-b}) = ae^{ax-b}$

WORKED EXAMPLE 5 Finding the derivative of an exponential function

Differentiate each exponential function.

a $y = e^{2x}$

b $y = e^{x^3+x-2}$

Steps	Working
a Use the rule $\frac{d}{dx}(e^{ax}) = ae^{ax}$.	$\frac{dy}{dx} = 2e^{2x}$
b **1** Use the chain rule: identify u and write y in terms of u.	Let $u = x^3 + x - 2$ so $y = e^u$.
2 Write $\frac{du}{dx}$ and $\frac{dy}{du}$ in terms of x.	$\frac{du}{dx} = 3x^2 + 1$ $\frac{dy}{du} = e^u = e^{x^3+x-2}$
3 Use $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.	$\frac{dy}{dx} = e^{x^3+x-2} \times (3x^2 + 1)$ $= (3x^2 + 1)e^{x^3+x-2}$

Chain, product and quotient rules

The chain rule: If $y = f(u)$ and $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.

or

If $y = f(g(x))$ then $y' = f'(g(x))g'(x)$.

The product rule: $\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$ or $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

The quotient rule: $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$ or $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

We can use the chain rule to create a formula for the derivative of e to the power of a function $f(x)$.

If $y = e^{f(x)}$, then $\frac{dy}{dx} = e^{f(x)} \times f'(x) = f'(x)e^{f(x)}$.

The chain rule for exponential functions

If $y = e^{f(x)}$, then $\frac{dy}{dx} = f'(x)e^{f(x)}$.

For example,

$$y = e^{3x^2-7x} \qquad \frac{dy}{dx} = (6x - 7)e^{3x^2-7x}$$

WORKED EXAMPLE 6 The chain and product rules

Find the derivative of each of the following functions.

a $y = (3e^{5x})^4$

b $y = x^2e^{-4x}$

Steps	Working
a **1** Use the chain rule: identify u and write y in terms of u.	Let $u = 3e^{5x}$ so $y = u^4$.
2 Obtain $\frac{du}{dx}$ and $\frac{dy}{du}$ in terms of x.	$\frac{du}{dx} = 3 \times 5e^{5x} = 15e^{5x}$ $\frac{dy}{du} = 4u^3 = 4(3e^{5x})^3$
3 Use $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.	$\frac{dy}{dx} = 4(3e^{5x})^3 \times (15e^{5x})$ $= 4 \times 27e^{15x} \times 15e^{5x}$ $= 1620e^{20x}$
b **1** Use the product rule: identify $f(x)$ and $g(x)$.	$f(x) = x^2$ and $g(x) = e^{-4x}$
2 Differentiate to obtain $f'(x)$ and $g'(x)$.	$f'(x) = 2x$, $g'(x) = -4e^{-4x}$
3 Write down the expression for $f'(x)g(x) + f(x)g'(x)$.	$\frac{dy}{dx} = f'(x)g(x) + f(x)g'(x)$ $= 2xe^{-4x} - 4x^2e^{-4x}$ $= 2xe^{-4x}(1 - 2x)$

WORKED EXAMPLE 7 The quotient rule

Find the value of the derivative of $f(x) = \frac{e^{3x}}{x^2 - 3x + 4}$ where $x = 2$.

Steps	Working
1 Let $\frac{u}{v} = \frac{e^{3x}}{x^2 - 3x + 4}$	$u = e^{3x}$ $\quad v = x^2 - 3x + 4$ $\frac{du}{dx} = 3e^{3x}$, $\quad \frac{dv}{dx} = 2x - 3$
2 Differentiate using the quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$	$f'(x) = \frac{3e^{3x}(x^2 - 3x + 4) - e^{3x}(2x - 3)}{(x^2 - 3x + 4)^2}$
3 Find $f'(2)$ by substituting $x = 2$ into $f'(x)$.	$f'(2) = \frac{3e^{3(2)}((2^2) - 3(2) + 4) - e^{3(2)}(2(2) - 3)}{(2^2 - 3(2) + 4)^2}$ $= \frac{5e^6}{4}$

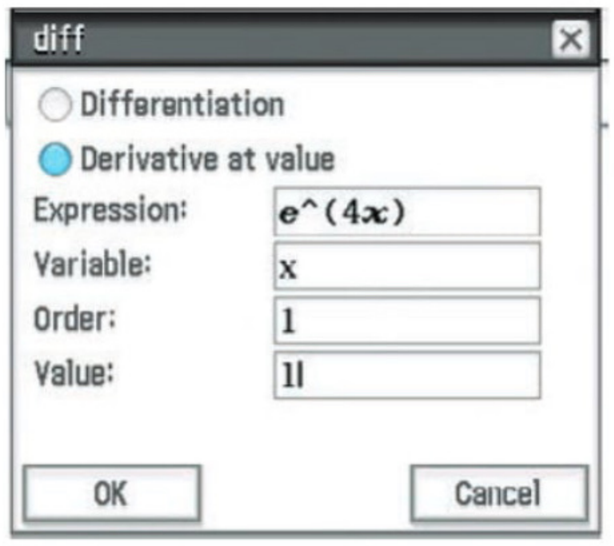
Exam hack

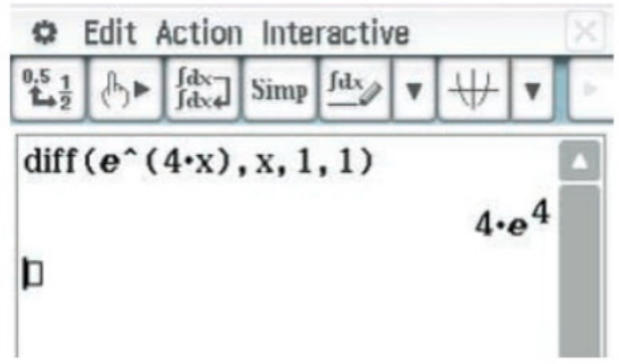
When finding $f'(x)$ for a particular x value, it's often faster to NOT simplify $f'(x)$ before substituting in the value of x.

USING CAS 1 Finding the first derivative of functions involving exponentials

Find the first derivative of $f(x) = e^{4x}$ at $x = 1$.

ClassPad





1 In **Main**, enter and highlight the expression.

2 Tap **Interactive > Calculation > diff**.

3 In the dialogue box tap **Derivative at value**.

4 In the **Value:** field, enter **1** and tap **OK**.

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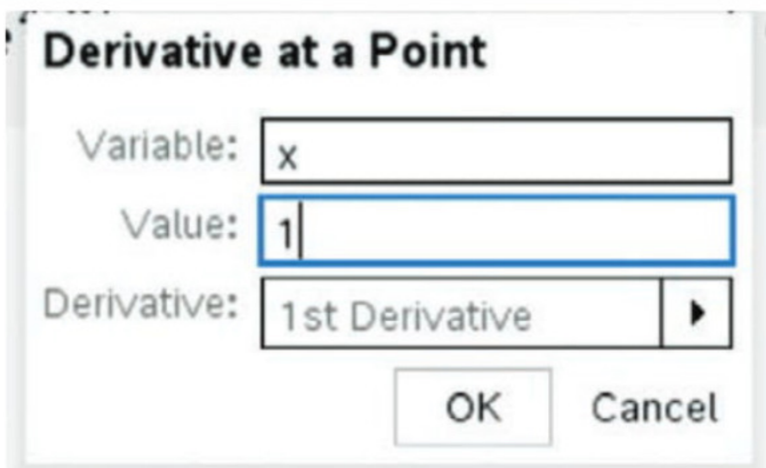
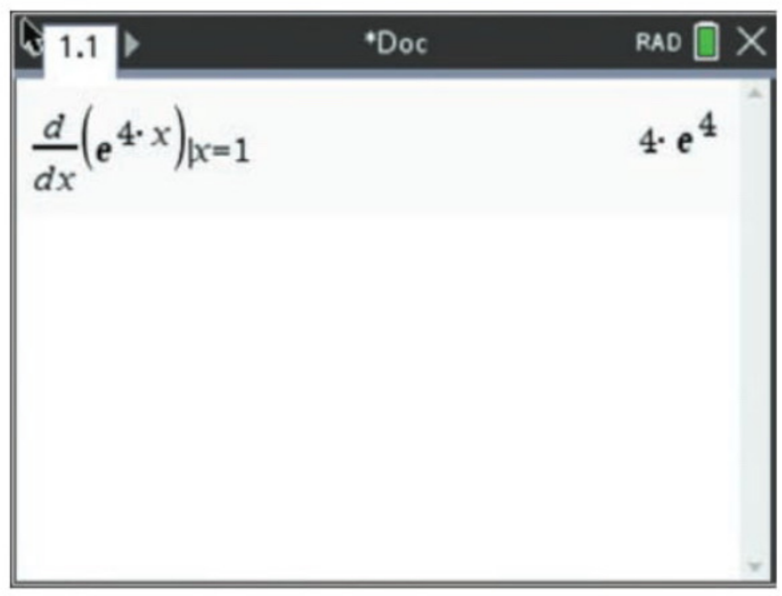




1 Press **menu > Calculus > Derivative at a Point**.

2 In the dialogue box **Value:** field, enter **1**.

3 In the template, enter the expression and press **enter**.

The first derivative at $x = 1$ is $4e^4$.

WORKED EXAMPLE 8 Finding the equation of the tangent to the curve

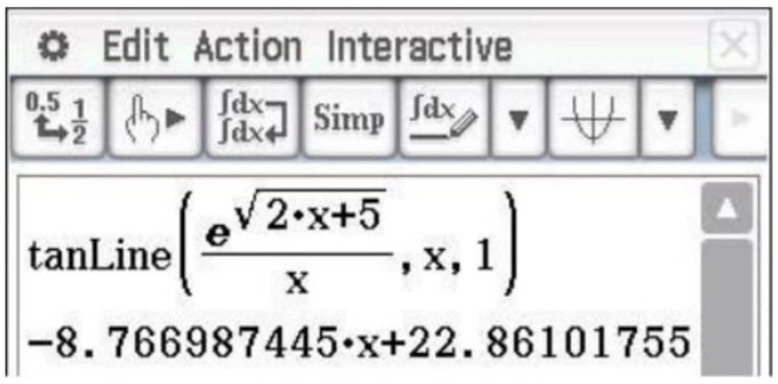
Find the equation of the tangent to the curve $f(x) = e^{2x+1}$ at $x = 1$.

Steps	Working
1 Find $f'(1)$.	$f'(x) = 2e^{2x+1}$ $f'(1) = 2e^{2(1)+1} = 2e^3$
2 Find $f(1)$.	$f(1) = e^{2(1)+1} = e^3$
3 Write the coordinates of the point on the tangent and the gradient.	Gradient $m = f'(1) = 2e^3$ $y = f(1) = e^3$ The point on the tangent is $(1, e^3)$.
4 Use the formula $y - y_1 = m(x - x_1)$ to find the equation of the tangent.	$y - e^3 = 2e^3(x - 1)$ $y - e^3 = 2e^3x - 2e^3$ $y = 2e^3x - e^3$

USING CAS 2 Finding the equation of the tangent to the curve

Find the approximate equation of the tangent , correct to two decimal places, to the curve $f(x) = \dfrac{e^{\sqrt{2x+5}}}{x}$ at $x = 1$.

ClassPad



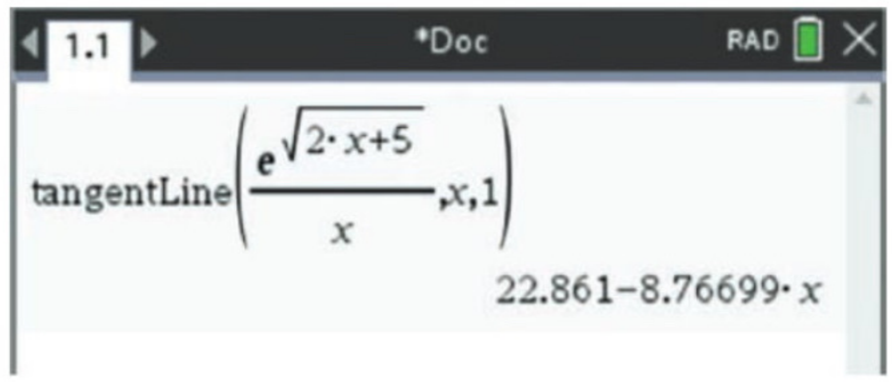
1 In **Main**, enter and highlight the expression $\dfrac{e^{\sqrt{2x+5}}}{x}$.

2 Tap **Interactive > Calculation > line > tanLine**.

3 In the dialogue box, **Point:** field, enter **1**.

4 Tap **OK**.

TI-Nspire



1 Press **menu > Calculus > Tangent Line**.

2 Enter the expression followed by **,x,1**.

3 Press **ctrl + enter** for the approximate solution.

The equation of the tangent is $y = -8.77x + 22.86$, correct to two decimal places.

WORKED EXAMPLE 9 Finding and describing the nature of a stationary point of an exponential function

Consider the function $y = e^{-2x^2}$.

Find

a $\dfrac{dy}{dx}$

b $\dfrac{d^2y}{dx^2}$

c the x value of the stationary point

d the nature of the stationary point.

Steps	Working
a Find the first derivative using the chain rule $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$.	$u = -2x^2$ $y = e^u$ $\dfrac{du}{dx} = -4x$ $\dfrac{dy}{du} = e^u = e^{-2x^2}$ $\dfrac{dy}{dx} = -4xe^{-2x^2}$
b Find the second derivative using the product rule: $\dfrac{dy}{dx}(uv) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$	$u = -4x$ $v = e^{-2x^2}$ $\dfrac{du}{dx} = -4$ $\dfrac{dv}{dx} = -4xe^{-2x^2}$ $\dfrac{d^2y}{dx^2} = 16x^2e^{-2x^2} - 4e^{-2x^2}$
c Solve the first derivative equal to zero to find the x value of the stationary points.	$\dfrac{dy}{dx} = 0$ $-4xe^{-2x^2} = 0$ $x = 0$ as $e^{-2x^2} > 0$

d Substitute $x = 0$ into the second derivative to find the nature of the stationary point.	$\frac{d^2y}{dx^2} = 16x^2e^{-2x^2} - 4e^{-2x^2}$ $\frac{d^2y}{dx^2} = 16(0)^2e^{-2(0)^2} - 4e^{-2(0)^2}$ $\frac{d^2y}{dx^2} = -4$ As $\frac{d^2y}{dx^2} < 0$ the stationary point at $x = 0$ is a local maximum.

Exponential modelling and differentiation

There are many natural phenomena that are modelled using simple exponential functions of the form $f(x) = Ae^{kx}$, where A and k are constants. In the case where the variable represents time, we write $f(t) = Ae^{kt}$.

WORKED EXAMPLE 10 **Using the natural exponential function and its derivative**

A metal cools down according to the formula $T = T_0e^{-0.1t}$, where T is the temperature difference between the metal and the surroundings in °C and t is time in minutes. The initial temperature is 228°C and the room is at 20°C.

a Evaluate T_0.

b Find, correct to one decimal place, the temperature difference after 20 minutes.

c What is the temperature after 20 minutes? Answer correct to one decimal place.

d Find, correct to one decimal place, the rate at which the metal is cooling after 20 minutes.

e Use the increments formula at $t = 20$ minutes to estimate the change in temperature for a 30 second change in time.

Steps	Working
a Find the initial temperature difference with the surroundings.	$T_0 = 228 - 20 = 208$°C
b 1 Substitute $t = 20$ into $T = T_0e^{-0.1t}$.	$T = 208e^{-0.1\times 20}$ $= 28.149\ldots$
2 State the result.	The temperature difference is about 28.1°C.
c Add the room temperature.	The temperature after 20 minutes is about $28.1 + 20 = 48.1$°C.
d 1 Find the derivative.	$T = 208e^{-0.1t}$ $\frac{dT}{dt} = 208 \times (-0.1e^{-0.1t})$ $= -20.8e^{-0.1t}$
2 Substitute $t = 20$.	rate of change $= -20.8e^{-0.1\times 20}$ $= -20.8e^{-2}$ $= -2.814\ldots$ The negative answer indicates cooling.
3 State the result.	After 20 minutes, the metal is cooling at about 2.8°C/min.

e Substitute $t = 20$ and $\delta t = 0.5$ into the formula $\delta T \approx \frac{dT}{dt} \times \delta t$.	$\delta T \approx \frac{dT}{dt} \times \delta t$ At $t = 20$, $\frac{dT}{dt} = -2.814$ and $\delta t = \frac{30}{60} = 0.5$. $\delta T \approx -2.814 \times 0.5 \approx -1.4°C$ The temperature difference decreases by approximately 1.4°C.

4.2

WORKED EXAMPLE 11 Using the derivative with exponential decay

Under exponential decay, the amount of radon-222 in milligrams that is present after t days is given by the function $f(t) = ae^{rt}$, where a and r are constants.

a If an initial amount of 100 mg decays to 84.11 mg after one day, show that the value of r is approximately -0.173.

b How much, correct to three decimal places, will 100 mg of radon-222 decay to in three days?

c At what rate is radon-222 decaying after one week? Give your answer to three decimal places.

Steps	Working
a **1** Write the formula.	$f(t) = ae^{rt}$
2 We need to find a and r. Substitute what we know when $t = 0$ (the initial condition).	When $t = 0$, $f(t) = 100$. $f(0) = ae^{0} = 100$ $a = 100$
3 Rewrite the formula with $a = 100$.	$f(t) = 100e^{rt}$
4 Substitute another condition to find r.	When $t = 1$, $f(t) = 84.11$. $f(1) = 100e^{r(1)} = 84.11$ $e^{r} = 0.8411$
5 Solve using CAS. *r* is negative because it is exponential decay.	$r \approx -0.173$
b **1** Write the formula with $r = -0.173$.	$f(t) = 100e^{-0.173t}$
2 Substitute $t = 3$.	$f(3) = 100e^{-0.173 \times 3}$ $= 59.512$
3 State the result.	59.512 mg of radon-222 remains.
c **1** Find $f'(t)$ for the rate of change.	$f(t) = 100e^{-0.173t}$ $f'(t) = 100 \times (-0.173)e^{-0.173t}$ $= -17.3e^{-0.173t}$
2 Substitute $t = 7$ for the rate after one week (7 days).	$f'(7) = -17.3e^{-0.173 \times 7}$ $= -17.3e^{-1.211}$ $= -5.154$
3 State the result.	After one week, radon-222 is decaying at a rate of 5.154 mg/day.

EXERCISE 4.2 Differentiating exponential functions

ANSWERS p. 394

Recap

1 The number of people, N, who have a virus at time t months is given by $N(t) = N_0e^{kt}$, where t is the number of months after the outbreak of the virus.

If the number is initially 50 and the number increases to 150 after one month, find

a the value of N_0

b the value of k, correct to four decimal places.

2 A radioactive element has a half-life of 20 minutes, meaning it takes 20 minutes to reach half its original size. Originally, there are 16.0×10^{20} atoms in the sample of the element. The decay is modelled by $N = N_0e^{-kt}$, where N_0 is the original number of atoms, N is the number of atoms present at time t minutes, and k is a constant. Find the value of k (correct to four decimal places).

Mastery

3 WORKED EXAMPLE 4 Using CAS find $\lim\limits_{h \to 0} \dfrac{a^h - 1}{h}$ for $a = 2.710, 2.711, 2.712, 2.713, 2.714, 2.715, 2.716, 2.717, 2.718, 2.719, 2.720$.

Write your answers correct to six decimal places and hence determine the best approximation for a for which $\lim\limits_{h \to 0} \dfrac{a^h - 1}{h} = 1$.

4 WORKED EXAMPLE 5 Differentiate each function.

a $y = 9e^x$ **b** $y = e^x + x^2$ **c** $y = (2e^x - 3)^6$

d $y = \dfrac{(e^x + e^{-x})^2}{e^x}$ **e** $y = e^{2x-1}$ **f** $y = e^{\sqrt{2x+4}}$

5 WORKED EXAMPLE 6 Find $f'(x)$ for each function.

a $f(x) = xe^x$ **b** $f(x) = (2x + 3)e^x$ **c** $f(x) = 5x^3e^x$

6 WORKED EXAMPLE 7 Find $g'(3)$ if $g(x) = \dfrac{e^x - 4}{\sqrt{e^x + 1}}$.

7 Using CAS 1 Find the derivative of each function.

a $y = \dfrac{e^x}{x^2}$ **b** $y = \dfrac{e^{6x}}{3x}$ **c** $y = \dfrac{2e^{5x}}{5x^3}$ **d** $y = \dfrac{x - 1}{e^x}$ **e** $y = \dfrac{e^x + 1}{e^{2x}}$

8 Given $y = e^{4x}(x^3 - 3x + 5)$, find $\dfrac{dy}{dx}$, for $x = -1$.

9 If $f(x) = \dfrac{xe^{3x} + 5}{x^2 + e}$, find $f'(2)$ correct to one decimal place.

10 $h(x) = 5x^2e^{3x} + e^x$. Find $h'(2)$.

11 Find the value of x such that the rate of change of xe^{2x-1} is $5e^3$.

12 WORKED EXAMPLE 8 Find the equation of the tangent to the curve $f(x) = \sqrt{e^x}$ at the point on the curve where $x = 1$.

13 Using CAS 2 Find the equation of the tangent to the curve $f(x) = \sqrt{e^{3x}}$ at the point where $x = 2$.

14 WORKED EXAMPLE 9 Consider the function $y = e^{2x^2-4x}$.

Find

a $\frac{dy}{dx}$

b $\frac{d^2y}{dx^2}$

c the x value of the stationary point

d the nature of the stationary point.

15 WORKED EXAMPLE 10 A study of swans in an area of Western Australia showed that their numbers were gradually increasing, with the number of swans N over t months given by $N(t) = 1100e^{0.025t}$.

a How many swans were there at the beginning of the study?

b How many swans were there after 5 months?

c At what rate was the number of swans increasing after 5 months?

d Use the increments formula at $t = 5$ months to estimate the change in the number of swans for a change in time of 0.1 month.

16 The area of rainforests is declining in a region of Queensland with the area A hectares over time t years given by $A(t) = 120\,000e^{-0.033t}$.

At what rate is the area of rainforest decreasing in this region after

a 2 years? **b** 15 years? **c** 40 years?

17 WORKED EXAMPLE 11 The amount of ${}^{226}_{88}Ra$, a common isotope of radium, in milligrams, that is present after t days is given by the function $R(t) = R_0e^{kt}$, where R_0 and k are constants. The initial amount of ${}^{226}_{88}Ra$ is 200 mg and this decays to 191.6 mg after one day.

Find

a R_0

b k, correct to three decimal places

c the rate at which ${}^{226}_{88}Ra$ is decaying after seven days.

Calculator-free

18 (7 marks)

a Differentiate x^3e^{2x} with respect to x. (2 marks)

b Let $f(x) = e^{x^2}$. Find $f'(3)$. (3 marks)

c Evaluate $f'(1)$, where $f(x) = e^{x^2-x+3}$. (2 marks)

19 © SCSA MM2016 Q3 (4 marks) Consider the function $f(x) = \frac{(x-1)^2}{e^x}$.

a Show that the first derivative is $f'(x) = \frac{-x^2 + 4x - 3}{e^x}$. (2 marks)

b Use your result from part **a** to explain why there are stationary points at $x = 1$ and $x = 3$. (2 marks)

20 (3 marks) Let $f(x) = e^x + k$, where k is a real number. The tangent to the graph of f at the point where $x = a$ passes through the point $(0,0)$. Find the value of k in terms of a.

Calculator-assumed

21 © SCSA MM2020 Q15 (9 marks) A chef needs to use an oven to boil 100 mL of water in five minutes for a new experimental recipe. The temperature of the water must reach 100°C in order to boil. The temperature, T, of 100 mL of water t minutes after being placed in an oven set to T_0°C can be modelled by the equation

$$T(t) = T_0 - 175e^{-0.07t}$$

In a preliminary experiment, the chef placed a 100 mL bowl of water into an oven that had been heated to $T_0 = 200$°C.

a What is the temperature of the water at the moment it is placed into the oven? (1 mark)

b What is the temperature of the water five minutes after being placed in the oven? (1 mark)

c What change could be made to the temperature at which the oven is set in order to achieve the five-minute boiling requirement? (2 marks)

Assume that T_0 is still 200°C.

d Determine the rate of increase in temperature of the water five minutes after being placed in the oven. Give your answer rounded to two decimal places. (2 marks)

e Explain what happens to the rate of change in the temperature of the water as time increases and how this relates to the temperature of the water. (3 marks)

22 © SCSA MM2018 Q14 (5 marks)

a The table below examines the values of $\frac{a^h - 1}{h}$ for various values of a as h approaches zero.

Copy and complete the table, rounding your values to five decimal places. (2 marks)

h	$a = 2.60$	$a = 2.70$	$a = 2.72$	$a = 2.80$
0.1	1.002 65		1.052 41	1.084 49
0.001	0.955 97	0.993 75		
0.00001	0.955 52			1.029 62

It can be shown that $\frac{d}{dx}(a^x) = a^h \lim_{h\to 0}\left(\frac{a^h - 1}{h}\right)$.

b What is the exact value of a for which $\frac{d}{dx}(a^x) = a^x$? Explain how the above definition and the table in part **a** support your answer. (3 marks)

4.3 Integrating exponential functions

We know that:

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(e^{ax}) = ae^{ax} \quad \text{(by the chain rule)}$$

We can reverse these rules to integrate exponential functions.

Also consider $y = e^{ax+b}$

$$\frac{dy}{dx} = ae^{ax+b} \quad \text{(by the chain rule)}$$

So, $\int ae^{ax+b}\,dx = e^{ax+b} + c$

and dividing both sides by a gives

$$\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + c$$

Video playlist
Integrating exponential functions

The integral of e^x

The **integral** of e^x is $e^x + c$:

$$\int e^x dx = e^x + c$$

The **integral** of e^{ax} is $\frac{1}{a}e^{ax} + c$:

$$\int e^{ax} dx = \frac{1}{a}e^{ax} + c$$

WORKED EXAMPLE 12 Finding the integral of an exponential function

Find the integral of each of the following.

a $\int e^{3-4x}\,dx$ **b** $\int \frac{e^{5x} + 3 + e^x}{4e^{3x}}\,dx$

Steps	Working
a Use $\int e^{ax+b}dx = \frac{1}{a}e^{ax+b} + c$ with $a = -4$, $b = 3$.	$\int e^{3-4x}dx = -\frac{1}{4}e^{3-4x} + c$
b **1** Separate the terms first.	$\int \frac{e^{5x} + 3 + e^x}{4e^{3x}}dx = \frac{1}{4}\int \frac{e^{5x}}{e^{3x}} + \frac{3}{e^{3x}} + \frac{e^x}{e^{3x}}dx$
2 Simplify by subtracting powers.	$= \frac{1}{4}\int e^{2x} + 3e^{-3x} + e^{-2x}\,dx$
3 Integrate each term using $\int e^{ax}dx = \frac{1}{a}e^{ax} + c$.	$= \frac{1}{4}\left(\frac{1}{2}e^{2x} + \frac{3}{-3}e^{-3x} + \frac{1}{-2}e^{-2x}\right) + c$ $= \frac{1}{8}e^{2x} - \frac{1}{4}e^{-3x} - \frac{1}{8}e^{-2x} + c$

WORKED EXAMPLE 13 Finding $f(x)$ given $f'(x)$ and a point

Find $f(x)$ if its gradient function is $f'(x) = 3e^{2x}$ and $f(0) = 8$.

Steps	Working
1 Integrate $f'(x)$ to find $f(x)$.	$f'(x) = 3e^{2x}$ $f(x) = \frac{3}{2}e^{2x} + c$
2 Substitute $x = 0, f(0) = 8$ to find c.	$8 = \frac{3}{2}e^{0} + c$ $8 - \frac{3}{2} = c$ $c = \frac{13}{2}$
3 Write the equation of the function.	$f(x) = \frac{3}{2}e^{2x} + \frac{13}{2}$

WORKED EXAMPLE 14 Evaluating definite integrals

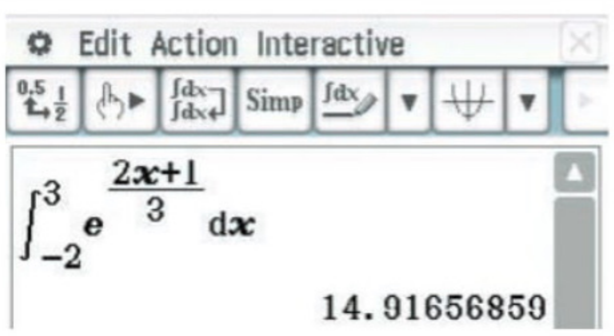
Evaluate $\int_{-2}^{2} 3e^{-\frac{x}{2}}dx$.

Steps	Working
1 Find the anti-derivative.	$\int_{-2}^{2} 3e^{-\frac{x}{2}}\,dx$ $= \left[\frac{3}{\frac{-1}{2}}e^{\frac{-x}{2}}\right]_{-2}^{2}$ $= \left[-6e^{\frac{-x}{2}}\right]_{-2}^{2}$
2 Evaluate the integral.	$= -6e^{-1} - -6e$ $= 6e - \frac{6}{e}$

USING CAS 3 Integrating exponential functions

Evaluate $\int_{-2}^{3} e^{\frac{2x+1}{3}}\,dx$ correct to three decimal places.

ClassPad



1. In **Main**, enter and highlight the expression $e^{\frac{2x+1}{3}}$.
2. Tap **Interactive > Calculation >** $\int$.
3. In the dialogue box, tap **Definite** to enter the lower and upper limits.
4. Tap **OK**.

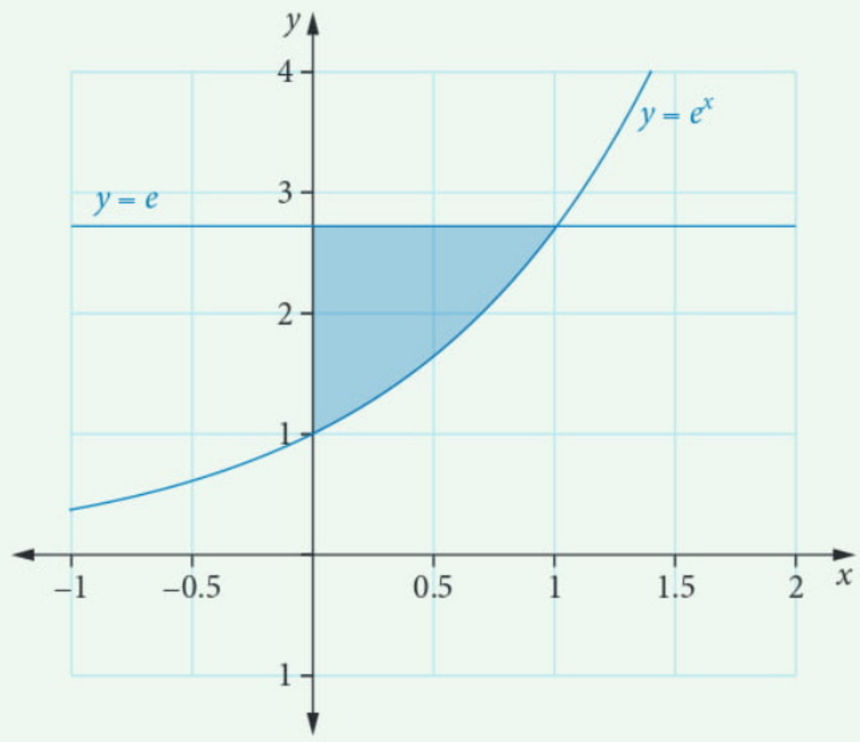
TI-Nspire

1. Press **menu > Calculus > Integral**.
2. Enter the lower limit, the upper limit and the expression.
3. Press **ctrl + enter** for the approximate solution.

The answer is 14.917, correct to three decimal places.

WORKED EXAMPLE 15 Finding areas using definite integrals

Find the bounded area between $y = e^x$, $y = e$ and the y-axis.



Steps	Working
1 Find the x value of the point of intersection.	$e^x = e$ $e^x = e^1$ $x = 1$
2 Write an integral equation to represent the area bounded by the functions. area = $\int$(upper curve − lower curve) dx	area = $\int_0^1 e - e^x \, dx$
3 Evaluate the integral.	area = $\left[ex - e^x\right]_0^1$ area = $e - e - \left(0 - e^0\right) = 1$ unit^2

EXERCISE 4.3 Integrating exponential functions

ANSWERS p. 395

Recap

1 Find $g'(2)$, correct to three decimal places, if $g(x) = e^{x^2} - 7$.

2 Find the gradient of the tangent to the graph of $y = e^{3x}$ at $x = 2$.

Mastery

3 WORKED EXAMPLE 12 Find each integral.

a $\int e^{-2x} \, dx$ **b** $\int 5e^{4x} \, dx$ **c** $\int e^{2x+1} \, dx$

d $\int \left(3e^{-2x} + e^{4x}\right) dx$ **e** $\int \frac{e^{4x} - 1}{e^x} \, dx$ **f** $\int \left(e^{3x} - e^{-3x}\right)^2 \, dx$

4 WORKED EXAMPLE 13 If the gradient function at a point (x, y) on a curve is given by $2e^{4x}$ and the curve passes through $(0, 2)$, find the equation of the curve.

5 If the gradient at a point (x, y) on a curve is given by $\frac{dy}{dx} = \frac{e^{3x} - 1}{e^x}$ and the curve passes through $(0, 11)$, find the equation of the curve.

6 A function $f(x)$ is such that $f'(x) = 5e^{-2x}$ and $f(3) = 2e$. Find $f(x)$.

7 WORKED EXAMPLE 14 Evaluate each definite integral.

a $\int_0^4 e^x \, dx$ **b** $\int_1^3 5e^x \, dx$ **c** $\int_2^4 \left(x^3 - e^x\right) dx$

8 Evaluate each integral, correct to two decimal places.

a $\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \left[\sin(3x) + e^{-6x}\right] dx$ **b** $\int_0^{\pi} \cos\left(\frac{x}{2}\right) + e^{3x} \, dx$

9 Using CAS 3 Evaluate $\int_0^3 5e^x - 2e^{3x} \, dx$ correct to one decimal place.

10 WORKED EXAMPLE 15 Find the bounded area between $y = e^{3x}$, $y = e^3$ and the y-axis.

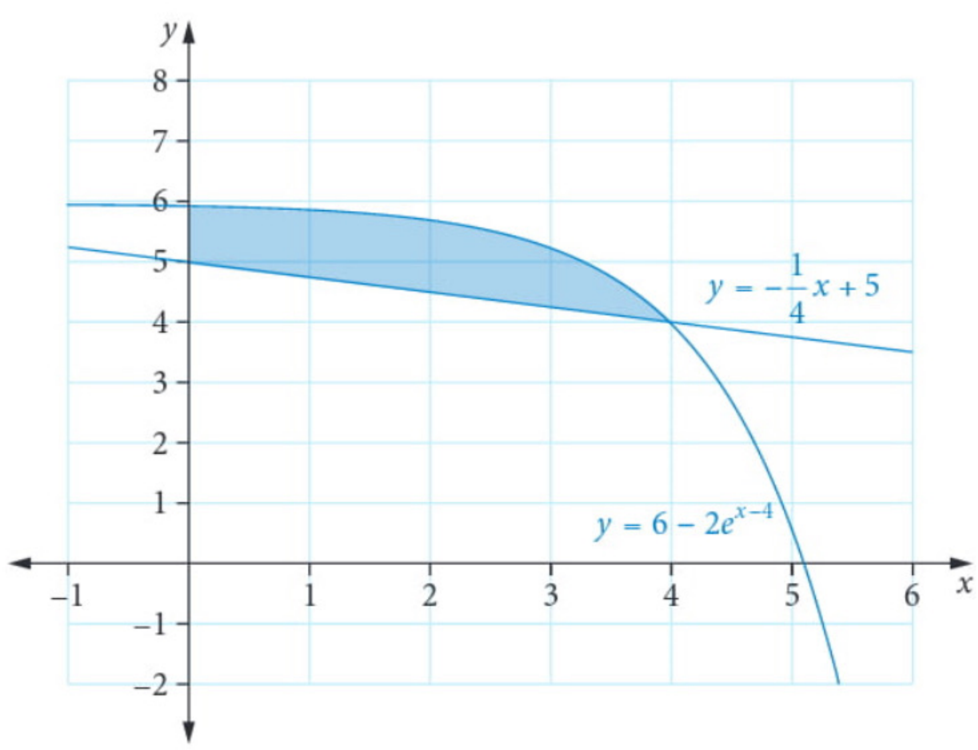
Calculator-free

11 © SCSA MM2016 Q2 (5 marks)

a Determine $\frac{d}{dx}\left(2xe^{2x}\right)$. (2 marks)

b Use your answer in part **a** to determine $\int 4xe^{2x} \, dx$. (3 marks)

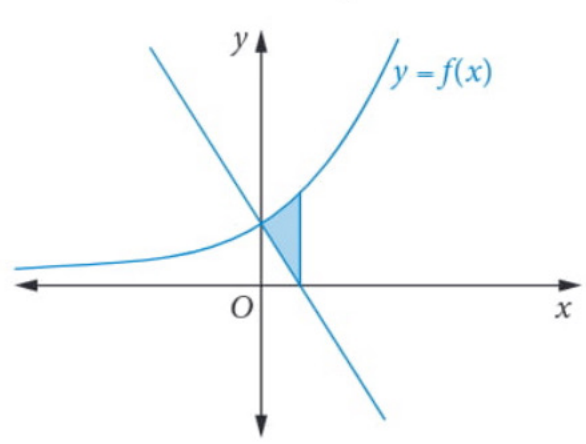
12 © SCSA MM2016 Q6 (4 marks) The graphs $y = 6 - 2e^{x-4}$ and $y = -\frac{1}{4}x + 5$ intersect at $x = 4$ for $x \geq 0$.



Determine the exact area between $y = 6 - 2e^{x-4}$, $y = -\frac{1}{4}x + 5$ and the y-axis for $x \geq 0$.

13 (5 marks) The graph of $f(x) = e^{\frac{x}{2}} + 1$ is shown.

The normal to the graph of f where it crosses the y-axis is also shown.

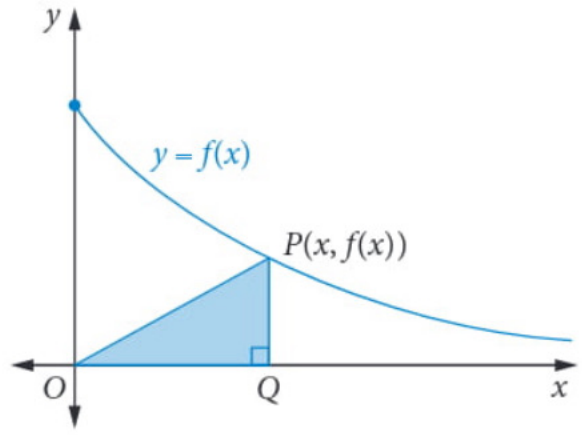


a Find the equation of the normal to the graph of f where it crosses the y-axis. (2 marks)

b Find the exact area of the shaded region. (3 marks)

14 (7 marks) Let $f(x) = 2e^{-\frac{x}{5}}$.

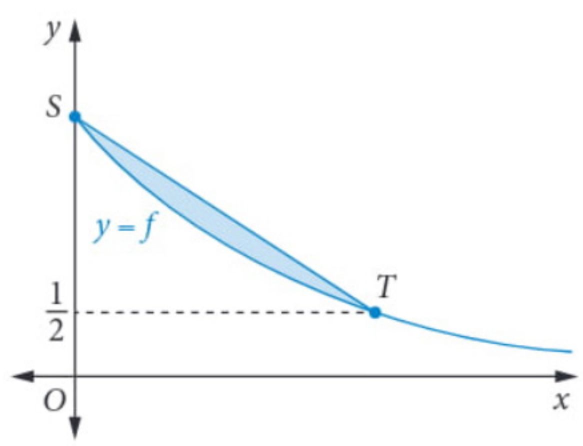
A right-angled triangle OQP has vertex O at the origin, vertex Q on the x-axis and vertex P on the graph of f, as shown. The coordinates of P are $(x, f(x))$.



a Find the area, A, of the triangle OQP in terms of x. (1 mark)

b Find the maximum area of triangle OQP and the value of x for which the maximum occurs. (3 marks)

c Let S be the point on the graph of f on the y-axis and let T be the point on the graph of f with the y-coordinate $\frac{1}{2}$. Find the area of the region bounded by the graph of f and the line segment ST. (3 marks)



15 © SCSA MM2017 Q15 (10 marks)

The volume $V(h)$ in cubic metres of a liquid in a large vessel depends on the height h (metres) of the liquid in the vessel and is given by

$$V(h) = \int_0^h e^{\left(-\frac{x^2}{100}\right)} dx, \; 0 \le h \le 15.$$

a Determine $\frac{dV}{dh}$ when the height is 0.5 m. (2 marks)

b What is the meaning of your answer to part **a**? (1 mark)

c The height of the liquid depends on time t (seconds) as follows:

$h(t) = 3t^2 - t + 4, \; t \ge 0$

i Determine $\frac{dh}{dt}$ when the height is 6 m. (2 marks)

ii Use the chain rule to determine $\frac{dV}{dt}$ when the height is 6 m. (2 marks)

iii Given the volume of the liquid at 2 seconds is 8.439 m^3, use the increments formula to estimate the volume 0.1 second later. (3 marks)

Video playlist
Differentiating trigonometric functions

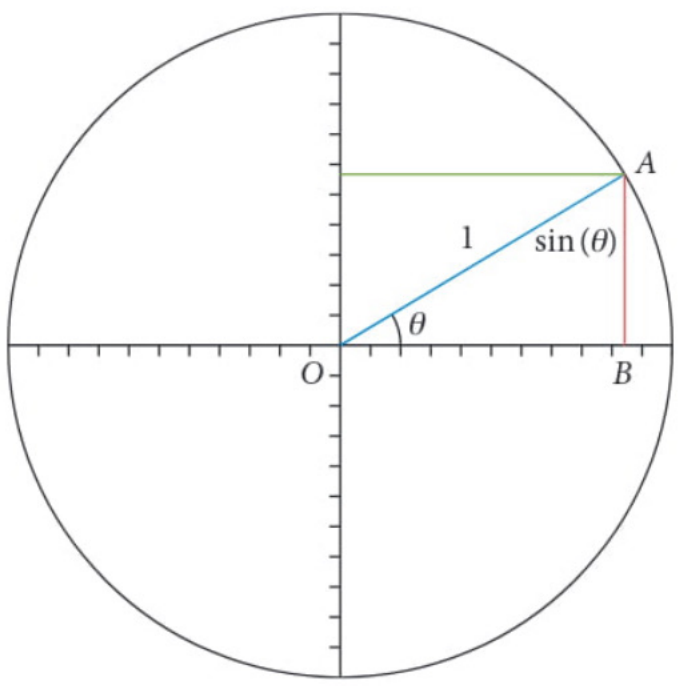
Worksheets
Further optimisation problems

Trigonometric functions and gradient

4.4 Differentiating trigonometric functions

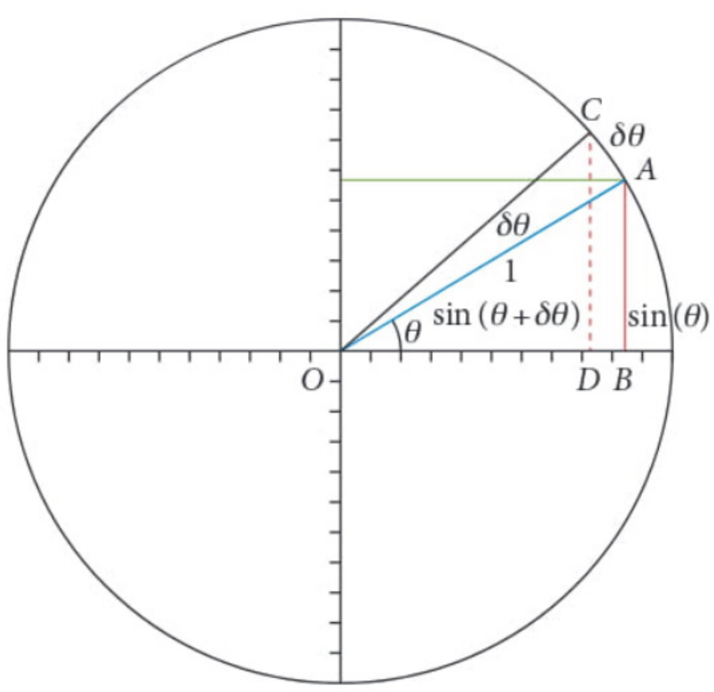
Derivatives of trigonometric functions

The geometric proof for the derivative of $y = \sin(\theta)$ uses the unit circle definition of $\sin(\theta)$ and the limit definition of the derivative.



In a unit circle, $\overline{AB} = \sin(\theta)$.

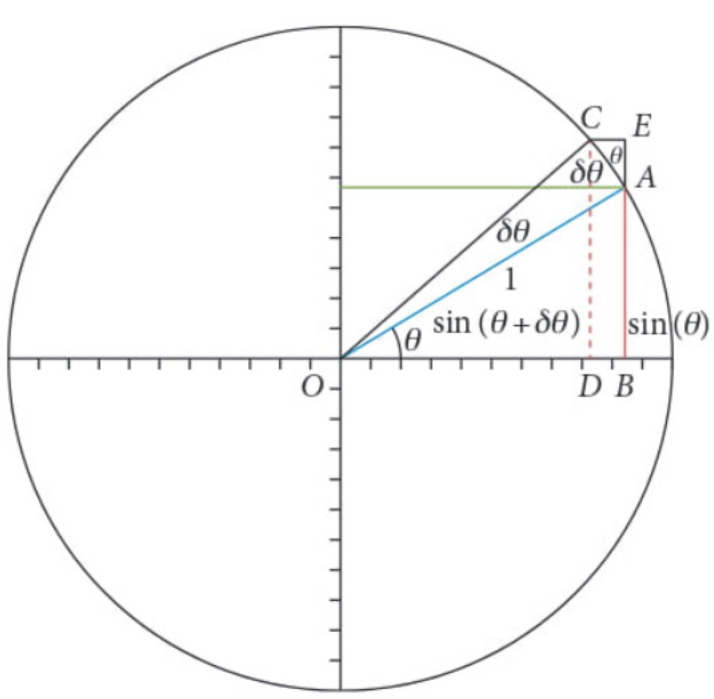
The limit definition for the derivative of $\sin(\theta)$ is $\dfrac{d(\sin(\theta))}{d\theta} = \lim\limits_{\delta\theta \to 0} \dfrac{\sin(\theta + \delta\theta) - \sin(\theta)}{\delta\theta}$.



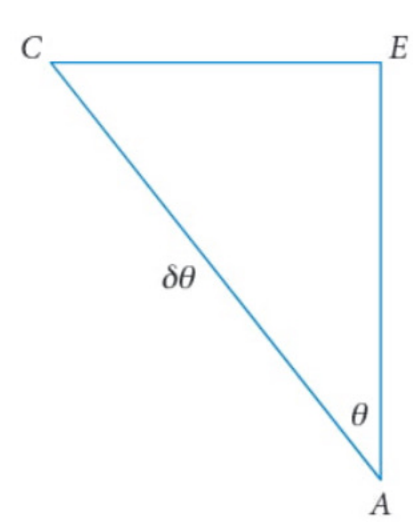
$\delta\theta$ is a small increase in the angle θ.

The arc length in a circle $= r \times \theta$, however, the unit circle has a radius of 1 so the length of arc AC is $\delta\theta$.

$\overline{CD} = \sin(\theta + \delta\theta)$ and $\lim\limits_{\delta\theta \to 0}(\text{arc } AC) = \overline{AC}$.



In triangle ACE



$\cos(\theta) = \dfrac{AE}{\delta\theta}$ and

$AE = \sin(\theta + \delta\theta) - \sin(\theta)$

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$$\cos(\theta) = \frac{\sin(\theta + \delta\theta) - \sin(\theta)}{\delta\theta} \text{ and } \frac{d(\sin(\theta))}{d\theta} = \lim_{\delta\theta \to 0} \frac{\sin(\theta + \delta\theta) - \sin(\theta)}{\delta\theta}$$

Therefore, $\frac{d(\sin(\theta))}{d\theta} = \lim_{\delta\theta \to 0} \cos(\theta) = \cos(\theta)$

Using the same diagram:

$CE = \cos(\theta) - \cos(\theta + \delta\theta)$ and $\sin(\theta) = \frac{CE}{\delta\theta}$

$$\frac{d(\cos(\theta))}{d\theta} = \lim_{\delta\theta \to 0} \frac{\cos(\theta + \delta\theta) - \cos(\theta)}{\delta\theta} = -\sin(\theta)$$

Derivatives of trigonometric functions

Trigonometric function	Derivative	Example
$y = \sin(ax - b)$	$\frac{dy}{dx} = a\cos(ax - b)$	$y = 3\sin(3x)$ $\frac{dy}{dx} = 3 \times 3\cos(3x)$ $\frac{dy}{dx} = 9\cos(3x)$
$y = \cos(ax - b)$	$\frac{dy}{dx} = -a\sin(ax - b)$	$y = 4\cos(2x)$ $\frac{dy}{dx} = -4 \times 2\sin(2x)$ $= -8\sin(2x)$

WORKED EXAMPLE 16 Using the product rule

Find the first derivative of $y = \sin(x)\cos(x)$.

Steps	Working
1 Use the product rule: identify u and v.	$y = \sin(x)\cos(x)$ $u = \sin(x)$ and $v = \cos(x)$
2 Differentiate to obtain $\frac{du}{dx}$ and $\frac{dv}{dx}$.	$\frac{du}{dx} = \cos(x)$, $\frac{dv}{dx} = -\sin(x)$
3 Substitute into the product rule $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$ and simplify. For the product rule, either function can be u or v. We can also let $u = \cos(x)$ and $v = \sin(x)$.	$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$ $\frac{dy}{dx} = \sin(x) \times -\sin(x) + \cos(x) \times \cos(x)$ $\frac{dy}{dx} = -\sin^2(x) + \cos^2(x)$

WORKED EXAMPLE 17 Using the quotient rule

Find $\frac{dy}{dx}$ for $y = \frac{2x^2 - x}{\sin(x)}$.

Steps	Working
1 Use the quotient rule: identify u and v where $y = \frac{u}{v}$.	$u = 2x^2 - x$ and $v = \sin(x)$
2 Differentiate to obtain $\frac{du}{dx}$ and $\frac{dv}{dx}$.	$\frac{du}{dx} = 4x - 1$, $\frac{dv}{dx} = \cos(x)$
3 Substitute into the quotient rule $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ and simplify. With the quotient rule, u must be the function in the numerator and v the function in the denominator.	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ $\frac{dy}{dx} = \frac{\sin(x) \times (4x - 1) - (2x^2 - x) \times \cos(x)}{(\sin x)^2}$ $\frac{dy}{dx} = \frac{4x\sin(x) - \sin(x) - 2x^2\cos(x) + x\cos(x)}{\sin^2(x)}$

WORKED EXAMPLE 18 Using the chain rule

Find the first derivative of $y = \sin(x^3 + 2x)$.

Steps	Working
1 Use the chain rule: identify u and write y in terms of u.	Let $u = x^3 + 2x$, so $y = \sin(u)$.
2 Find $\frac{du}{dx}$ and $\frac{dy}{du}$ in terms of x.	$\frac{du}{dx} = 3x^2 + 2$ $\frac{dy}{du} = \cos(u) = \cos(x^3 + 2x)$
3 Use $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.	$\frac{dy}{dx} = \cos(x^3 + 2x) \times (3x^2 + 2)$ $= (3x^2 + 2)\cos(x^3 + 2x)$

WORKED EXAMPLE 19 Combining the rules

Find $\frac{dy}{dx}$ for $y = x^3 \sin(x^2)$.

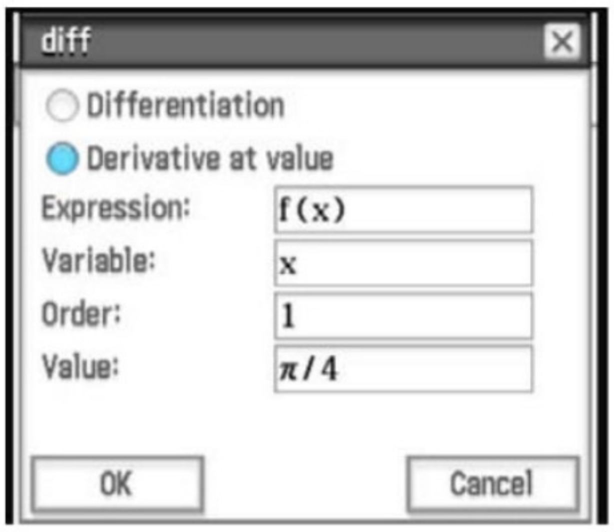
Steps	Working
1 Identify the differentiation rules to be used.	The product rule and chain rule are needed.
2 Identify u and v in the product rule.	$u = x^3$ $\quad v = \sin(x^2)$
3 Obtain $\frac{du}{dx}$ and $\frac{dv}{dx}$ using the chain rule. The chain rule for $\frac{d(\sin(f(x)))}{dx}$ is $f'(x)\cos(f(x))$.	$\frac{du}{dx} = 3x^2$ $\quad \frac{dv}{dx} = 2x\cos(x^2)$
4 Use the product rule.	$\frac{dy}{dx} = \sin(x^2) \times 3x^2 + x^3 \times 2x\cos(x^2)$ $= 3x^2\sin(x^2) + 2x^4\cos(x^2)$

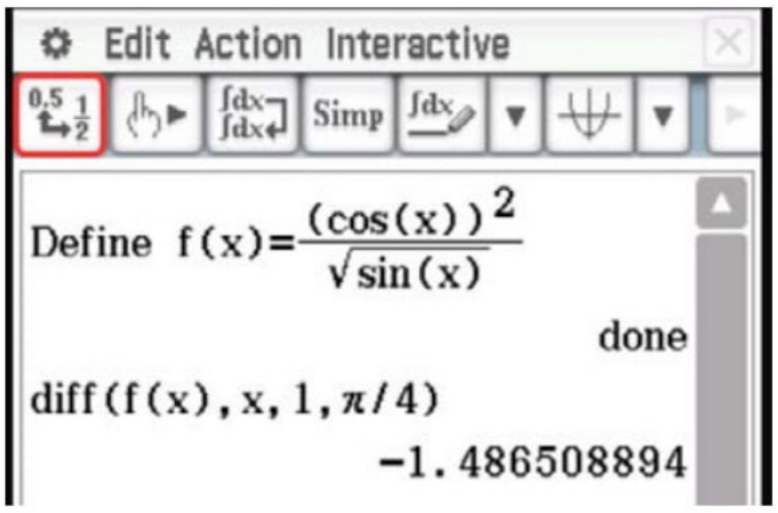
4.4

USING CAS 4 Finding the derivative of a trigonometric function at a point

For $f(x) = \dfrac{\cos^2(x)}{\sqrt{\sin(x)}}$, calculate the value of $f'\left(\dfrac{\pi}{4}\right)$ correct to three decimal places.

ClassPad





1 Define **f(x)** = $\dfrac{\cos^2(x)}{\sqrt{\sin(x)}}$. Note that $\cos^2(x)$ needs to be entered as $(\cos(x))^2$.

2 Highlight **f(x)** and tap **Interactive > Calculation > diff**.

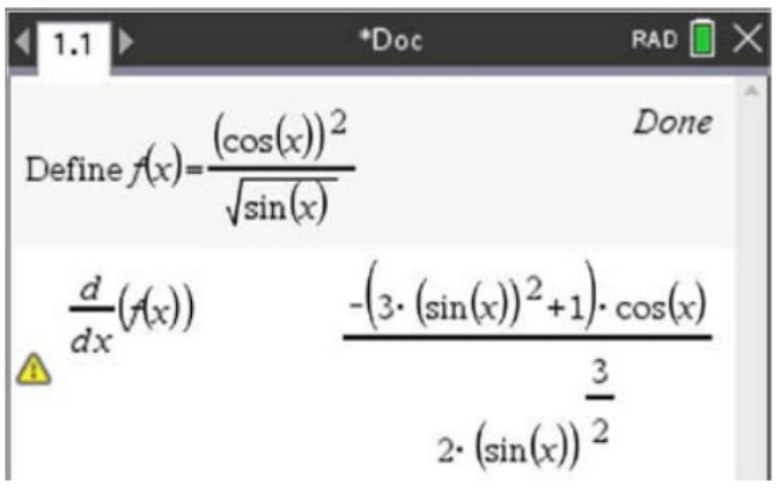
3 In the dialogue box, tap **Derivative at value**.

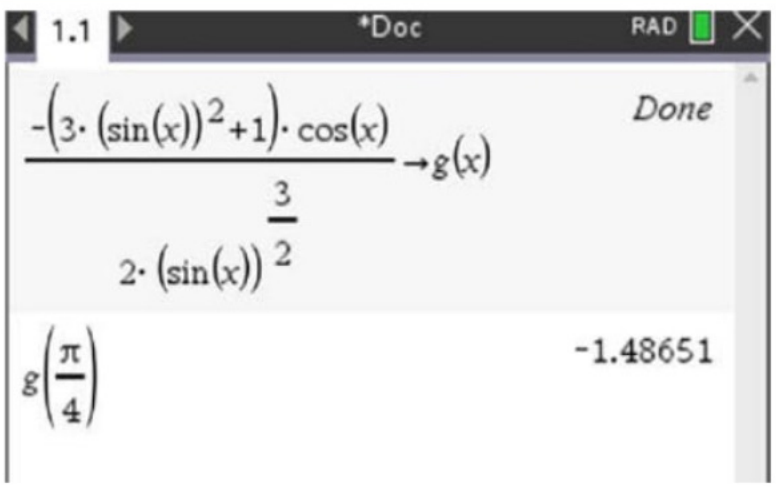
4 Enter Value: $\dfrac{\pi}{4}$.

5 Tap **OK** and the answer will be shown.

6 Change to decimals if required using **Convert** or by tapping **Decimal** at the bottom of the screen.

TI-Nspire





1 Define **f(x)** as shown above. Note that $\cos^2(x)$ needs to be entered as $(\cos(x))^2$.

2 Find the derivative of **f(x)**.

3 Press **ctrl + var** to store the derivative as **g(x)**.

4 Enter $g\left(\dfrac{\pi}{4}\right)$ and press **ctrl + enter** for the approximate answer.

$f'\left(\dfrac{\pi}{4}\right) \approx -1.487$, to three decimal places.

Straight line motion

displacement, x, at time t: $x(t)$

velocity, v, at time t: $v(t) = \dfrac{dx}{dt}$

acceleration, a, at time t: $a(t) = \dfrac{dv}{dt} = \dfrac{d^2x}{dt^2}$

WORKED EXAMPLE 20 Straight line motion

An oscillating spring moves so that its end is x cm from the point P at time t seconds, where $x = 2\sin(4t)$.

a Find an equation for the velocity of the spring.

b What is the initial velocity of the spring?

c What is the maximum acceleration?

Steps	Working
a velocity $= \dfrac{dx}{dt}$	$x = 2\sin(4t)$ $v = \dfrac{dx}{dt} = 2 \times 4\cos(4t)$ $= 8\cos(4t)$
b Initial velocity occurs when $t = 0$.	When $t = 0$, $v = 8\cos(4 \times 0)$ $= 8 \times 1$ $= 8$ cm/s
c Differentiate the velocity equation to obtain acceleration, a. Find the maximum acceleration.	$v = 8\cos(4t)$ $a = \dfrac{dx}{dt} = -32\sin(4t)$ The maximum acceleration is 32 cm/s^2 when $\sin(4t) = -1$.

WORKED EXAMPLE 21 Finding optimal solutions

The effectiveness (f) of an insect repellent, measured over a 12-hour period, is given by the function $f(t) = e^t \sin\left(\dfrac{\pi t}{12}\right)$, where t is the number of hours after the repellent is applied.

a Show that $f'(t) = e^t\left(\sin\left(\dfrac{\pi t}{12}\right) + \dfrac{\pi}{12}\cos\left(\dfrac{\pi t}{12}\right)\right)$.

b Show that the maximum effectiveness occurs when $\tan\left(\dfrac{\pi t}{12}\right) = -\dfrac{\pi}{12}$.

c Determine the approximate coordinates of the local maximum correct to nearest integer.

Steps	Working
a Use the product rule to find the first derivative.	$u = e^t$, $\quad v = \sin\left(\dfrac{\pi t}{12}\right)$ $\dfrac{du}{dt} = e^t \quad \dfrac{dv}{dt} = \dfrac{\pi}{12}\cos\left(\dfrac{\pi t}{12}\right)$ $f'(t) = e^t\sin\left(\dfrac{\pi t}{12}\right) + e^t \times \dfrac{\pi}{12}\cos\left(\dfrac{\pi t}{12}\right)$ $f'(t) = e^t\left(\sin\left(\dfrac{\pi t}{12}\right) + \dfrac{\pi}{12}\cos\left(\dfrac{\pi t}{12}\right)\right)$

b Solve $f'(t) = 0$.

Stationary point at $f'(t) = 0$.

$$e^t\left(\sin\left(\frac{\pi t}{12}\right) + \frac{\pi}{12}\cos\left(\frac{\pi t}{12}\right)\right) = 0$$

$$\sin\left(\frac{\pi t}{12}\right) + \frac{\pi}{12}\cos\left(\frac{\pi t}{12}\right) = 0$$

$$\sin\left(\frac{\pi t}{12}\right) = -\frac{\pi}{12}\cos\left(\frac{\pi t}{12}\right)$$

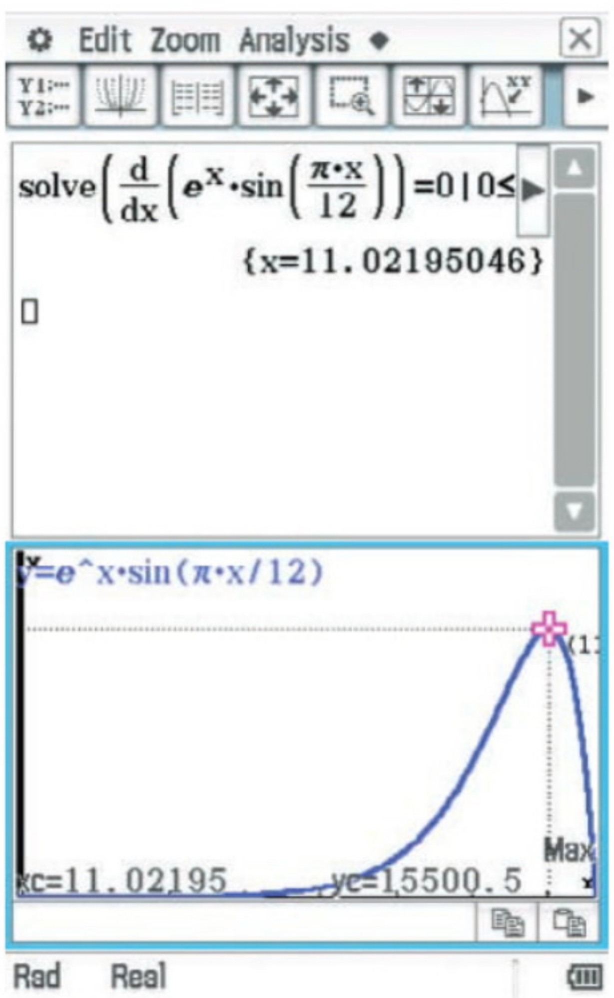
$$\frac{\sin\left(\frac{\pi t}{12}\right)}{\cos\left(\frac{\pi t}{12}\right)} = -\frac{\pi}{12}$$

$$\tan\left(\frac{\pi t}{12}\right) = -\frac{\pi}{12}$$

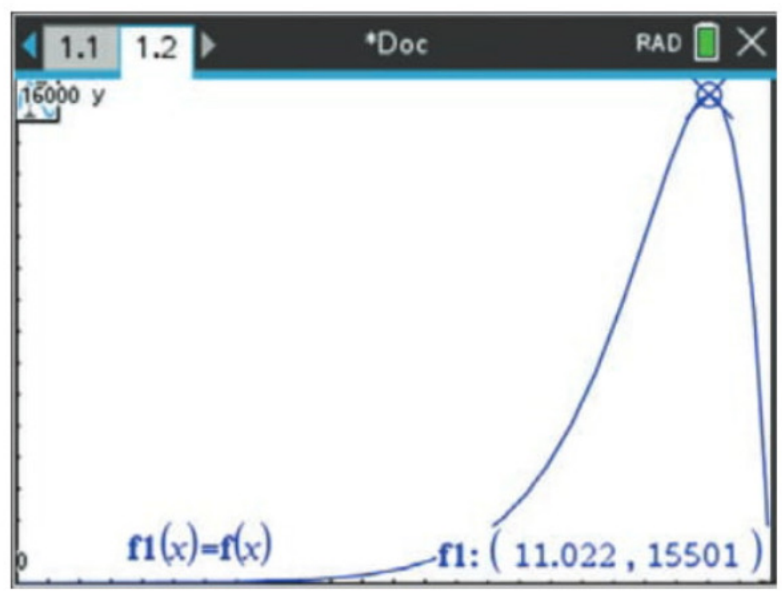
c Use CAS to find the coordinates of the stationary point.

Coordinates of the local maximum: (11, 15 501)

ClassPad



TI-Nspire



Maximum rate of change

The **maximum rate of increase or decrease** of a function is where the function is increasing or decreasing most rapidly, where the gradient of its graph is steepest in the positive and negative directions. We can identify these points by graphing the derivative function and locating its maximum and minimum points. These show where the function has its maximum rate of increase and decrease, respectively.

Finding the maximum rate of increase or decrease of a function $y = f(x)$

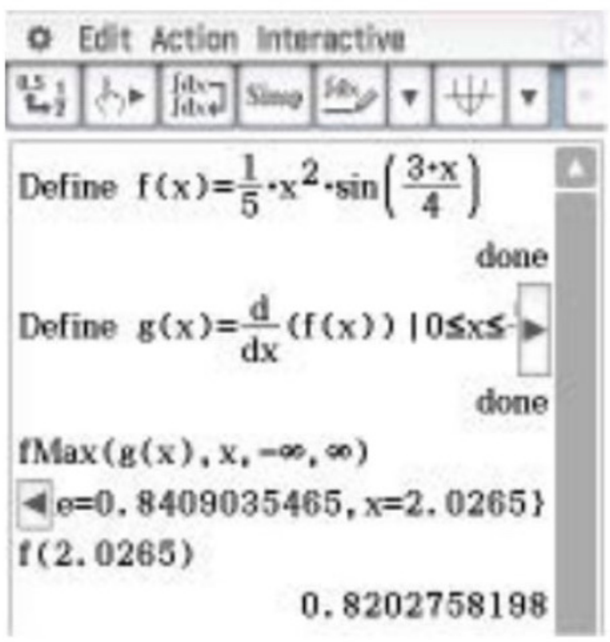
1 Graph $y = f(x)$ and $y = f'(x)$ for the given domain.
2 Find the local maximum and minimum points of $y = f'(x)$.
3 Substitute the x values of these points into $y = f(x)$ to find the points where the function has its maximum rate of increase and decrease, respectively.
4 Substitute the x values into $y = f'(x)$ to find the maximum rate of increase and decrease, respectively.

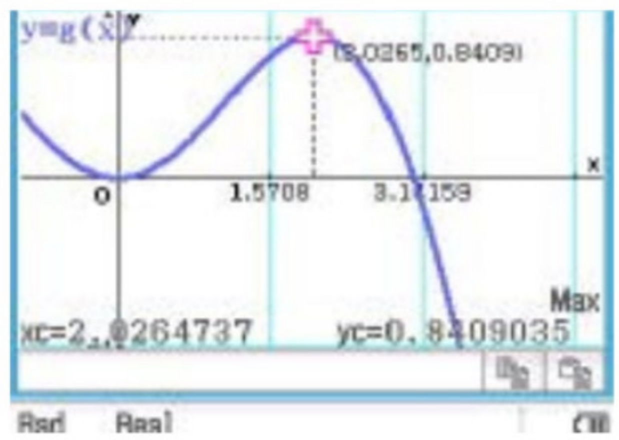
USING CAS 5 Maximum rate of change

A section of a rollercoaster track is described by the function $f(x) = \frac{1}{5}x^2 \sin\left(\frac{3x}{4}\right)$, where $0 \le x \le \frac{5\pi}{4}$.

Find, correct to two decimal places, the coordinates of the point on the track at which the rollercoaster is experiencing its maximum rate of increase.

ClassPad



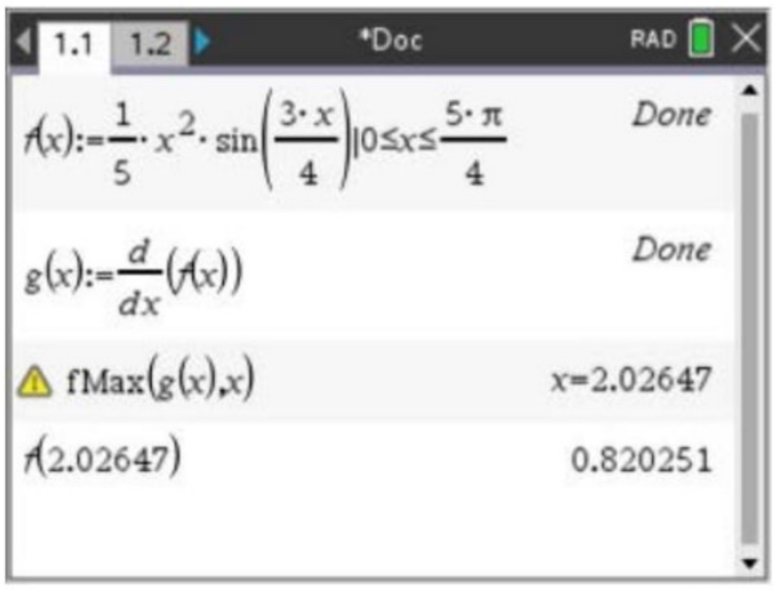


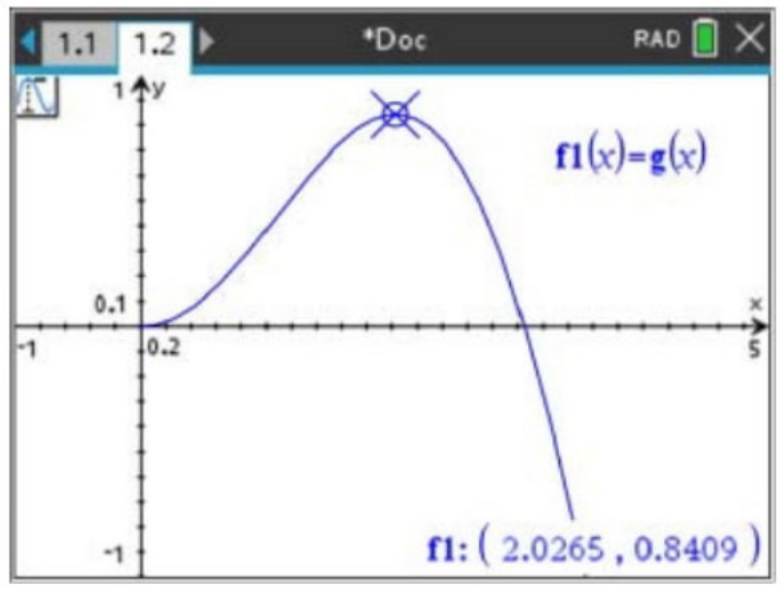
The point (2.0265…,0.8409…) is on the first derivative function. We must substitute $x = 2.0265$ into $f(x)$ to find the actual coordinates of the point of inflection.

1 Define $f(x)$ as shown above.
2 Define $g(x)$ as the derivative of $f(x)$.
3 Use **Math3** to include the domain $0 \le x \le \frac{5\pi}{4}$.
4 Highlight $g(x)$ and tap **Interactive > Calculation > fMin/fMax > fMax**.
5 The maximum value will be displayed.
6 Copy the answer into **f(x)** to find the y coordinate of the maximum point.
7 To confirm the result, graph $g(x)$ in split screen in **Main**. Include the domain $0 \le x \le \frac{5\pi}{4}$.
8 Adjust the window settings to suit.
9 Tap **Analysis > G-Solve > Max**.
10 The coordinates of the maximum point on $g(x)$ will be displayed.

The coordinates of the point of inflection experiencing the maximum positive rate of increase are (2.03, 0.82).

TI-Nspire





1. Define $f(x)$ as shown above.
2. Define $g(x)$ as the derivative of $f(x)$.
3. Press **catalog** and scroll down to **fMax**.
4. Enter **g(x),x** to find the x coordinate of the maximum point.
5. Copy the answer into **f(x)** to find the y coordinate of the maximum point.
6. To confirm the result, graph $g(x)$.
7. Adjust the window settings to suit.
8. Press **menu > Trace > Graph trace**.
9. Scroll along the graph until maximum appears.

The coordinates of the point of inflection experiencing the maximum positive rate of increase are (2.03, 0.82).

EXERCISE 4.4 Differentiating trigonometric functions

ANSWERS p. 395

Recap

1 Find $\int \left(e^{2x} - e^{-2x}\right)^2 dx$.

2 Evaluate $\int_0^3 4e^{2x} dx$.

Mastery

3 WORKED EXAMPLE 16 Differentiate $y = x^2 \sin(x)$.

4 WORKED EXAMPLE 17 Find $\frac{dy}{dx}$ for $y = \frac{5\cos(3x)}{\sin(2x)}$.

5 WORKED EXAMPLE 18 Find $\frac{dy}{dx}$ if $y = \sin\left(4x^5 - x\right)$.

6 WORKED EXAMPLE 19 Differentiate $y = 4x^2 \cos\left(x^3\right)$.

7 Using CAS 4

a For $f(x) = \sqrt{\cos(x)}$, find $f'\left(\frac{\pi}{6}\right)$ correct to three decimal places.

b Given $f'(c) = -\frac{\sqrt{6}}{4}$, find the value of c, correct to three decimal places if $0 \le c \le \frac{\pi}{2}$.

8 WORKED EXAMPLE 20 A particle moves so that its displacement x is given by the function $x(t) = 3\cos\left(\frac{t}{2}\right)$, $0 \le t \le 2\pi$, where x is in centimetres and t is in seconds.

- **a** Find an equation for the velocity of the particle.
- **b** Find an equation for the acceleration of the particle.
- **c** Find the time(s) when the particle is at $x = 0$.
- **d** Determine the velocity and acceleration when $x = 0$.
- **e** Find the time(s) at which the particle has the greatest acceleration.

9 WORKED EXAMPLE 21 The effectiveness (f) of an insect repellent, measured over a 3-hour period, is given by the function $f(t) = e^{\frac{t}{2}} \sin\left(\frac{\pi t}{3}\right)$, where t is the number of hours after the repellent is applied.

- **a** Show $f'(t) = e^{\frac{t}{2}}\left(\frac{1}{2}\sin\left(\frac{\pi t}{3}\right) + \frac{\pi}{3}\cos\left(\frac{\pi t}{3}\right)\right)$.
- **b** Show the maximum effectiveness occurs when $\tan\left(\frac{\pi t}{3}\right) = -\frac{2\pi}{3}$.
- **c** Determine the approximate coordinates of the local maximum, correct to one decimal place.

10 Using CAS 5

- **a** Consider the function $f(x) = x\sin\left(\frac{x}{4}\right)$, where $0 \le x \le 10$. Find, correct to two decimal places, the coordinates of the point on $f(x)$ where the function has its maximum rate of increase.
- **b** A waterslide track is described by the function $y = \frac{4\sin(x)}{x}$, where $\frac{\pi}{4} \le x \le 3\pi$. Determine the points on the waterslide where the rate of increase is maximum and where the rate of increase is minimum.

Calculator-free

11 © SCSA MM2020 Q2 (4 marks) If $h(x) = \frac{e^{-x}}{\cos x}$, then evaluate $h'(\pi)$.

12 (3 marks)

- **a** Let $f(x) = x\sin(x)$. Find $f'(x)$. (1 mark)
- **b** If $f(x) = \frac{x}{\sin(x)}$, find $f'\left(\frac{\pi}{2}\right)$. (2 marks)

13 (2 marks) If $g(x) = x^2\sin(2x)$, find $g'\left(\frac{\pi}{6}\right)$.

14 (2 marks) P is a point on the curve defined by $y = 4 + 2\cos(3x)$, where $0 \le x \le \frac{\pi}{3}$, and the tangent at P is parallel to the line $y = 1 - 6x$.

- **a** Find the coordinates of P. (1 mark)
- **b** Determine the equation of the tangent line at P. (1 mark)

15 © SCSA MM2016 Q4 (8 marks) The displacement x micrometres at time t seconds of a magnetic particle on a long straight superconductor is given by the rule $x = 5\sin 3t$.

a Determine the velocity of the particle when $t = \frac{\pi}{2}$. (3 marks)

b Determine the rate of change of the velocity when $t = \frac{\pi}{2}$. (3 marks)

Let v = velocity of the particle at t seconds.

c Determine $\int_0^{\frac{\pi}{2}} \frac{dv}{dt}\, dt$. (2 marks)

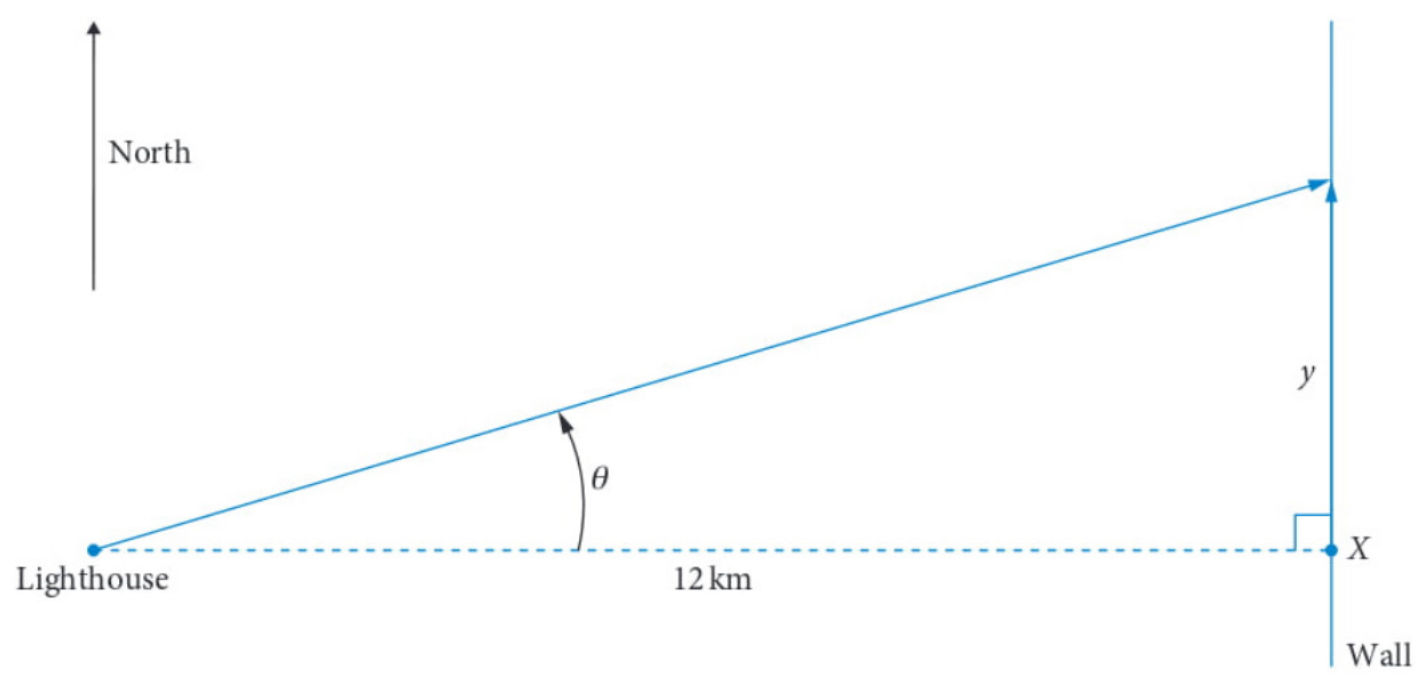
16 (3 marks) The position of a particle is given by $x = t\cos(2t)$, where x metres is the particle's position from a fixed point after t seconds. Find the acceleration of the particle after $\frac{\pi}{4}$ seconds.

Calculator assumed

17 © SCSA MM2016 Q21 (6 marks) A lighthouse is situated 12 km away from the shoreline, opposite point X as seen in the diagram below. A long brick wall is placed along the shoreline and at night the light from the lighthouse can be seen moving along this wall.

Let y = displacement of light on the wall from point X and θ = angle of the rotating light from the lighthouse.

The light is revolving anticlockwise at a uniform rate of three revolutions per minute $\left(\frac{d\theta}{dt} = 6\pi \text{ radians/minute}\right)$.



a Show that $\frac{dy}{d\theta} = \frac{12}{\cos^2\theta}$. (3 marks)

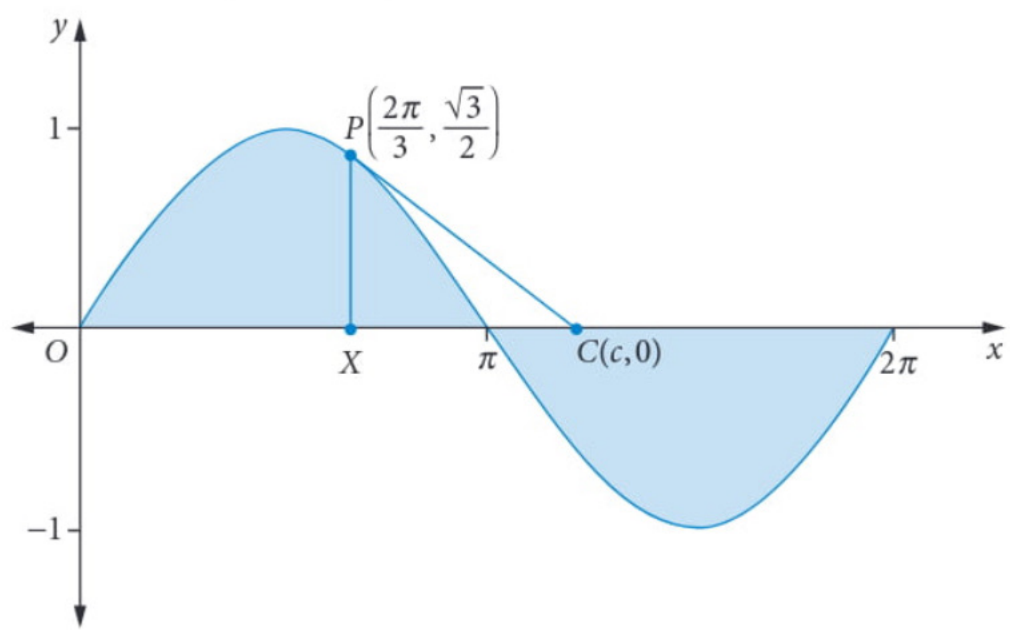
b Determine the velocity, in kilometres per minute, of the light on the wall when the light is 5 km north of point X. (3 marks)

$\left(\text{Hint: } \frac{dy}{dt} = \frac{dy}{d\theta} \times \frac{d\theta}{dt}\right)$

18 © SCSA MM2017 Q10 (3 marks) Use the quotient rule to show that $\frac{d}{dx}\tan(x) = \frac{1}{\cos^2(x)}$.

19 (2 marks) Trigg is designing a garden that is to be built on flat ground. In his initial plans, he draws the graph of $y = \sin(x)$ for $0 \le x \le 2\pi$ and decides that the garden beds will have the shape of the shaded regions shown in the diagram below. He includes a garden path, which is shown as line segment PC.

The line through points $P\left(\frac{2\pi}{3}, \frac{\sqrt{3}}{2}\right)$ and $C(c, 0)$ is a tangent to the graph of $y = \sin(x)$ at point P.



a Find $\frac{dy}{dx}$ when $x = \frac{2\pi}{3}$. (1 mark)

b Show that the value of c is $\sqrt{3} + \frac{2\pi}{3}$. (1 mark)

4.5 Integrals of trigonometric functions

Video playlist Integrals of trigonometric functions

Worksheets Finding indefinite integrals 1; Finding indefinite integrals 2; Finding definite integrals

Recall this table of derivatives of trigonometric functions:

Trigonometric function	Derivative
$y = \sin(ax - b)$	$\frac{dy}{dx} = a\cos(ax - b)$
$y = \cos(ax - b)$	$\frac{dy}{dx} = -a\sin(ax - b)$

We use these derivatives to obtain the anti-derivatives of trigonometric functions:

$$\frac{d}{dx}(\sin(ax - b)) = a\cos(ax - b)$$
$$\int \frac{d}{dx}(\sin(ax - b))\,dx = \int a\cos(ax - b)\,dx$$
$$\int \cos(ax - b)\,dx = \frac{1}{a}\sin(ax - b) + c$$

$$\frac{d}{dx}(\cos(ax - b)) = -a\sin(ax - b)$$
$$\int \frac{d}{dx}(\cos(ax - b))\,dx = \int -a\sin(ax - b)\,dx$$
$$\int \sin(ax - b)\,dx = -\frac{1}{a}\cos(ax - b) + c$$

Integrals of trigonometric functions

$$\int \sin(ax - b)\,dx = -\frac{1}{a}\cos(ax - b) + c$$

$$\int \cos(ax - b)\,dx = \frac{1}{a}\sin(ax - b) + c$$

4.5

WORKED EXAMPLE 22	Finding indefinite and definite integrals
a $\int 6\cos(2x) + 12\sin(3x)\,dx$ **b** $\int_0^{\pi} 4\cos(2x)\,dx$	
Steps	**Working**
a 1 Anti-differentiate each term.	$\int 6\cos(2x)dx = 6 \times \frac{1}{2}\sin(2x) = 3\sin(2x)$ $\int 12\sin(3x)\,dx = -12 \times \frac{1}{3}\cos 3x = -4\cos(3x)$
2 Combine the answers and include the constant of integration.	The anti-derivative of $6\cos(2x) + 12\sin(3x)$ is $F(x) = 3\sin(2x) - 4\cos(3x) + c$.
b Integrate the function and substitute the limits of integration.	$\int_0^{\pi} 4\cos(2x)\,dx = [2\sin(2x)]_0^{\pi}$ $= 2\sin(2\pi) - 2\sin(0)$ $= 0 - 0$ $= 0$

Anti-differentiation by recognition

WORKED EXAMPLE 23	Anti-differentiation by recognition
Given that $\frac{d}{dx}(x\cos(x)) = \cos(x) - x\sin(x)$, write the expression for $\int 6x\sin(x)\,dx$.	
Steps	**Working**
1 Transpose the equation so that $x\sin(x)$ is the subject.	$x\sin(x) = \cos(x) - \frac{d}{dx}(x\cos(x))$
2 Integrate every term in the equation.	$\int x\sin(x)\,dx = \int \cos(x)\,dx - \int \frac{d}{dx}(x\cos(x))\,dx$
3 Simplify the right-hand side of the equation.	$\int x\sin(x)\,dx = \sin(x) - x\cos(x)$
4 Multiple both sides by 6 and then add the constant 'c'.	$\int 6x\sin(x)\,dx = 6\sin(x) - 6x\cos(x) + c$

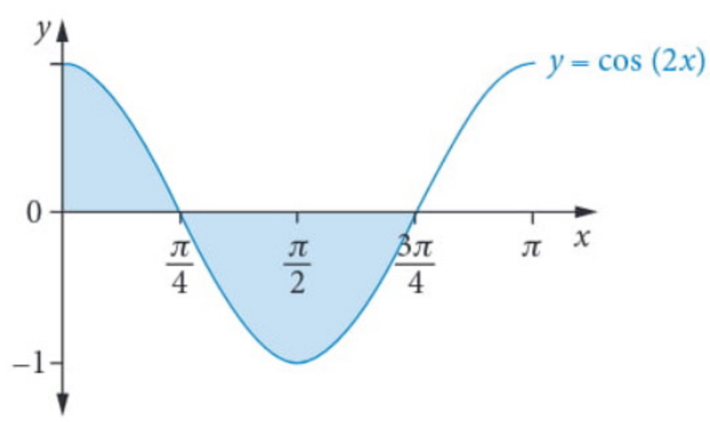
WORKED EXAMPLE 24	Anti-differentiating to find the constant of integration
The derivative of a function is $f'(x) = -4\sin(3x)$. If $f\left(\frac{\pi}{18}\right) = -\frac{2\sqrt{3}}{3}$, determine the function $f(x)$.	
Steps	**Working**
1 Anti-differentiate $f'(x) = -4\sin(3x)$.	$f(x) = \int -4\sin(3x)\,dx$ $= \frac{4}{3}\cos(3x) + c$

2 Find the value of c using $f\left(\frac{\pi}{18}\right) = -\frac{2\sqrt{3}}{3}$.	When $x = \frac{\pi}{18}$, $f(x) = -\frac{2\sqrt{3}}{3}$. $$f(x) = \frac{4}{3}\cos(3x) + c$$ $$-\frac{2\sqrt{3}}{3} = \frac{4}{3}\cos\left(3 \times \frac{\pi}{18}\right) + c$$ $$= \frac{4}{3}\cos\left(\frac{\pi}{6}\right) + c$$ $$= \frac{4}{3} \times \frac{\sqrt{3}}{2} + c$$ $$= \frac{2\sqrt{3}}{3} + c$$ $$\therefore c = -\frac{4\sqrt{3}}{3}$$
3 State the answer.	$f(x) = \frac{4}{3}\cos(3x) - \frac{4\sqrt{3}}{3}$

WORKED EXAMPLE 25 Finding areas for trigonometric functions by symmetry

a Sketch the graph of the curve $y = \cos(2x)$ in the interval $0 \le x \le \pi$.

b Calculate the area bounded by the curve, the y-axis and the x-axis in the interval $0 \le x \le \frac{3\pi}{4}$.

Steps	**Working**
a Sketch the graph in the interval $[0, \pi]$.	
b 1 The total area is three times the area above the x-axis. Write an integral equation for the area and evaluate.	Area $= 3\int_0^{\frac{\pi}{4}} \cos(2x)\, dx$
2 Calculate the total area.	$$= 3\left[\frac{1}{2}\sin(2x)\right]_0^{\frac{\pi}{4}}$$ $$= 3\left[\frac{1}{2}\sin\left(\frac{\pi}{2}\right) - \frac{1}{2}\sin(0)\right]$$ $$= 3\left[\frac{1}{2} - 0\right]$$ $$= \frac{3}{2}\text{ units}^2$$

9780170477536

Straight line motion

4.5

WORKED EXAMPLE 26 Straight line motion using integration

It takes an elevator 20 seconds to ascend from the ground floor of a building to the fifth floor. The velocity of the elevator during its ascent is given by

$$v(t) = \frac{7\pi}{20}\sin\left(\frac{\pi t}{20}\right)\text{m/s}.$$

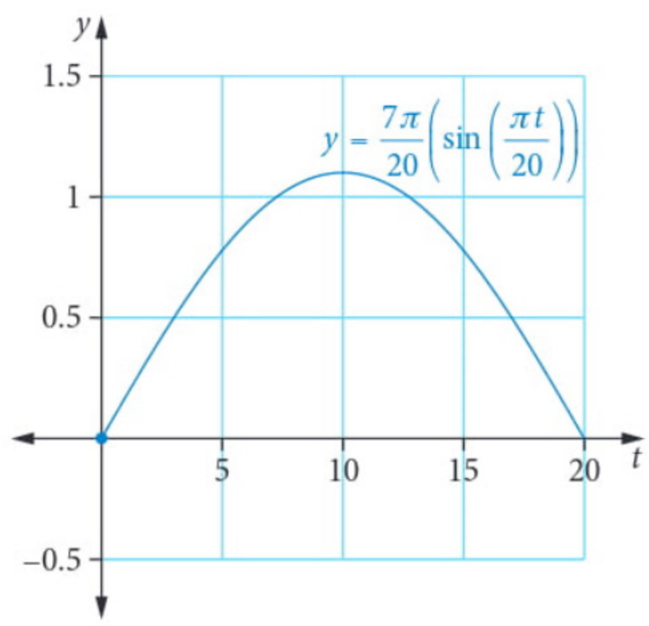
The velocity, v, is measured in metres per second, and the time, t, is measured in seconds.

Find

a the acceleration of the elevator in terms of t

b $\int_0^{\frac{10}{3}} v'(t)\,dt$

c the displacement function of the elevator if the ground floor has displacement zero metres.

Steps	Working
a 1 Graph the function.	
2 Find the derivative $a(t) = \frac{dv}{dt}$.	$a(t) = \frac{dv}{dt}$ $= \frac{7\pi}{20} \times \frac{\pi}{20}\cos\left(\frac{\pi t}{20}\right)$ $= \frac{7\pi^2}{400}\cos\left(\frac{\pi t}{20}\right)\text{m/s}^2$
b Substitute into $\int_a^b f'(x)\,dx = f(b) - f(a)$.	$\int_0^{\frac{10}{3}} v'(t)\,dt = v\left(\frac{10}{3}\right) - v(0)$ $= \frac{7\pi}{20}\sin\left(\frac{\pi}{20} \times \frac{10}{3}\right) - \frac{7\pi}{20}\sin(0)$ $= \frac{7\pi}{20}\sin\left(\frac{\pi}{6}\right)$ $= \frac{7\pi}{40}\text{ m/s}$

c **1** Find the integral of $v(t)$:

$x(t) = \int v(t)\, dt + c$.

$$x(t) = \int \frac{7\pi}{20} \sin\left(\frac{\pi t}{20}\right) dt + c$$
$$= \frac{7\pi}{20} \int \sin\left(\frac{\pi t}{20}\right) dt + c$$
$$= \frac{7\pi}{20} \times \frac{20}{\pi} \times -\cos\left(\frac{\pi t}{20}\right) + c$$
$$= -7\cos\left(\frac{\pi t}{20}\right) + c$$

2 Substitute $x = 0$, $t = 0$ to find the constant c.

$$x(0) = 0$$
$$0 = -7\cos(0) + c$$
$$c = 7$$
$$x(t) = 7 - 7\cos\left(\frac{\pi t}{20}\right)$$

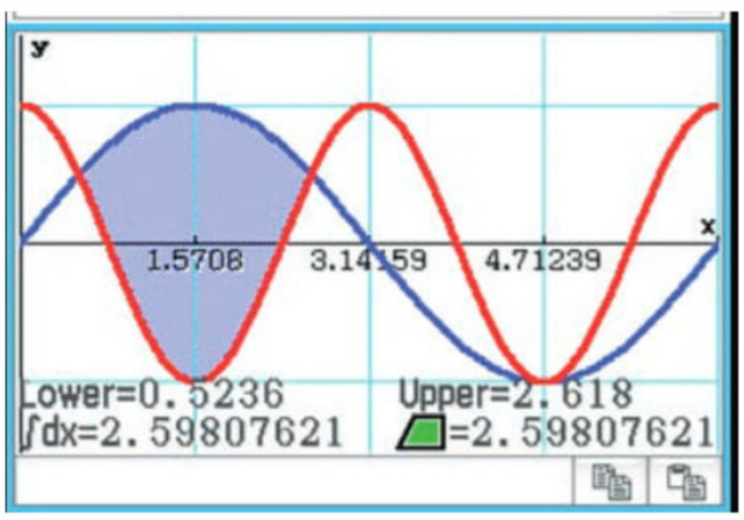
Area between two curves

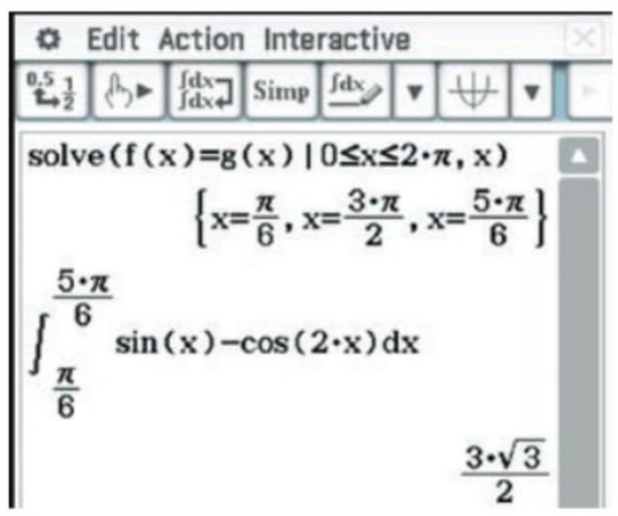
The area between the curves with equations $y = f(x)$ and $y = g(x)$, where $f(x) > g(x)$, in the interval $a \le x \le b$ is found using area $= \int_a^b [f(x) - g(x)]\, dx$.

USING CAS 6 Area between two curves

Find, correct to three decimal places, the area of the region bounded by the curves $y = \sin(x)$ and $y = \cos(2x)$ in the interval $0 \le x \le \pi$.

ClassPad

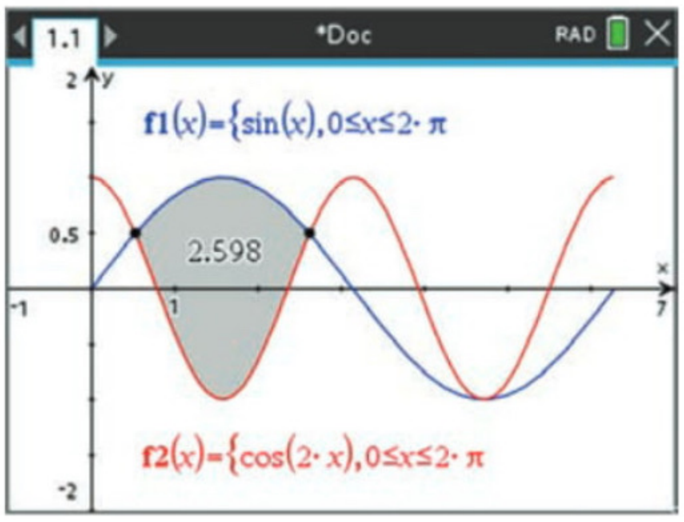


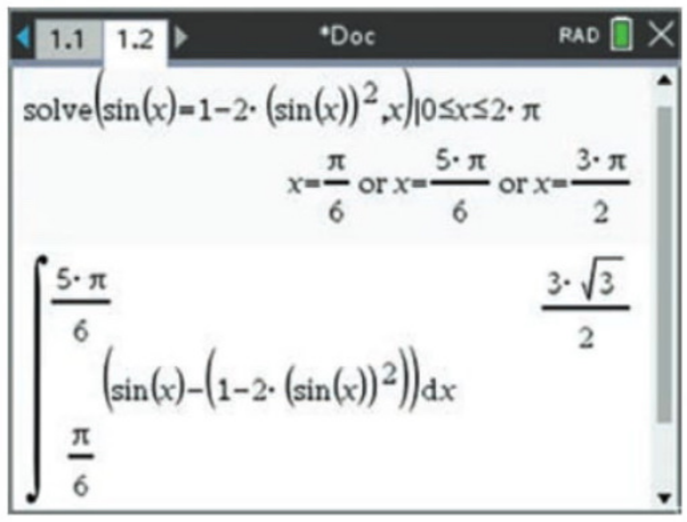


1 In **Main**, define **f(x)** = sin (*x*) and **g(x)** = cos (2*x*).
2 Open the graph screen, highlight and drag down $f(x)$ and $g(x)$.
3 Adjust the window settings to suit $0 \le x \le 2\pi$.
4 Tap **Analysis > G-Solve > Integral > ∫dx Intersection**.
5 When the first point of intersection is displayed, press **EXE**, then tap the right arrow.
6 When the second point of intersection is displayed, press **EXE**.
7 The shaded area between the curves and the value will be displayed.
8 To find the exact area, solve the equation **sin (x)=cos (2x)** or **f(x)=g(x)**, over the domain **0≤x≤2**π.
9 Highlight **sin (x)–cos (2x)** or **f(x)–g(x)** and tap **Interactive > Calculation > ∫**.
10 In the dialogue box, tap **Definite**.
11 Enter the two smaller solutions as the lower and upper limits, and enter the expression as shown above.
12 The exact area will be displayed.

The area is 2.598 units^2, correct to three decimal places.

TI-Nspire





1. Graph **f1(x)** and **f2(x)** as shown above.
2. Adjust the window settings to suit.
3. Press **menu > Analyze Graph > Bounded Area**.
4. When prompted for the **lower bound**, click on the first point of intersection.
5. When prompted for the **upper bound**, click on the second point of intersection.
6. The area will be displayed.
7. To find the exact area, you need to use the identity $\cos(2x) = 1 - 2\sin^2(x)$.
8. Solve the equation as shown above over the domain $0 \leq x \leq 2\pi$.
9. Press **menu > Calculus > Integral**.
10. Enter the two smaller solutions as the lower and upper limits, and enter the expression as shown above.
11. The exact area will be displayed.

The area is 2.598 units^2, correct to three decimal places.

WACE QUESTION ANALYSIS

© SCSA MM2021 Q12 Calculator-assumed (15 marks)

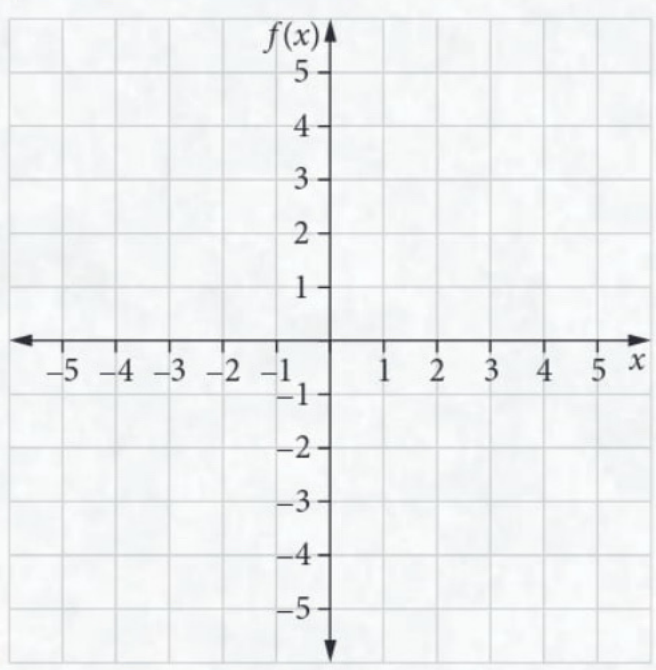
Let $f(x) = x^2 e^x$.

a Show that $f'(x) = xe^x(x + 2)$. (2 marks)

b Use calculus to determine all the stationary points of $f(x)$ and determine their nature. (7 marks)

c Determine the coordinates of any points of inflection. (2 marks)

d Copy the axes below and hence sketch the graph of $f(x)$, clearly indicating the location of all stationary points and points of inflection. (4 marks)



4.5

Video WACE question analysis: Applying the exponential and trigonometric functions

Reading the question

- Highlight the type of answer required in each part. This may be coordinates, a first derivative, a second derivative or a graph.
- The 'show that' in part **a** must include all the working for the derivative and cannot be done by CAS.
- Highlight all the details that must be labelled on the graph.

Thinking about the question

- This question requires knowledge of the product and chain rules for differentiation.
- You need to be able to find a second derivative and know how to use it to determine the nature of a stationary point.
- You also need to use a second derivative to determine the coordinates of points of inflection.
- The sketch graph drawn must show the coordinates of all stationary points and points of inflection.

Worked solution (✓ = 1 mark)

a Use the product rule to differentiate.

$u = x^2$ $\quad\quad$ $v = e^x$

$\frac{du}{dx} = 2x$ $\quad\quad$ $\frac{dv}{dx} = e^x$

$f'(x) = 2xe^x + x^2e^x = xe^x(x+2)$

differentiates using product rule ✓

factorises correctly ✓

b $\boldsymbol{f'(x) = 0}$ ✓

$xe^x(x+2) = 0$

stationary points at $x = \mathbf{0}$ and $x = \mathbf{-2}$ ✓

At $x = 0, f(0) = 0$ $\quad\quad$ and at $x = -2$ $\quad f(-2) = (-2)2e^{-2} = \frac{4}{e^2} \approx 0.54.$

coordinates of stationary points $\boldsymbol{(0,0)}$ ✓ and $\boldsymbol{\left(-2, \frac{4}{e^2}\right)}$ ✓

Nature of stationary points

$f''(x) = 2xe^x + x^2e^x + 2e^x + 2xe^x$

$\boldsymbol{f''(x) = e^x(x^2 + 4x + 2)}$ ✓

At $x = 0, \quad f''(0) = 2 > 0$

Therefore, $(0,0)$ is a local **minimum.** ✓

At $x = -2, \quad f''(-2) = -2e^{-2} < 0$

Therefore, $\left(-2, \frac{4}{e^2}\right)$ is a local **maximum**. ✓

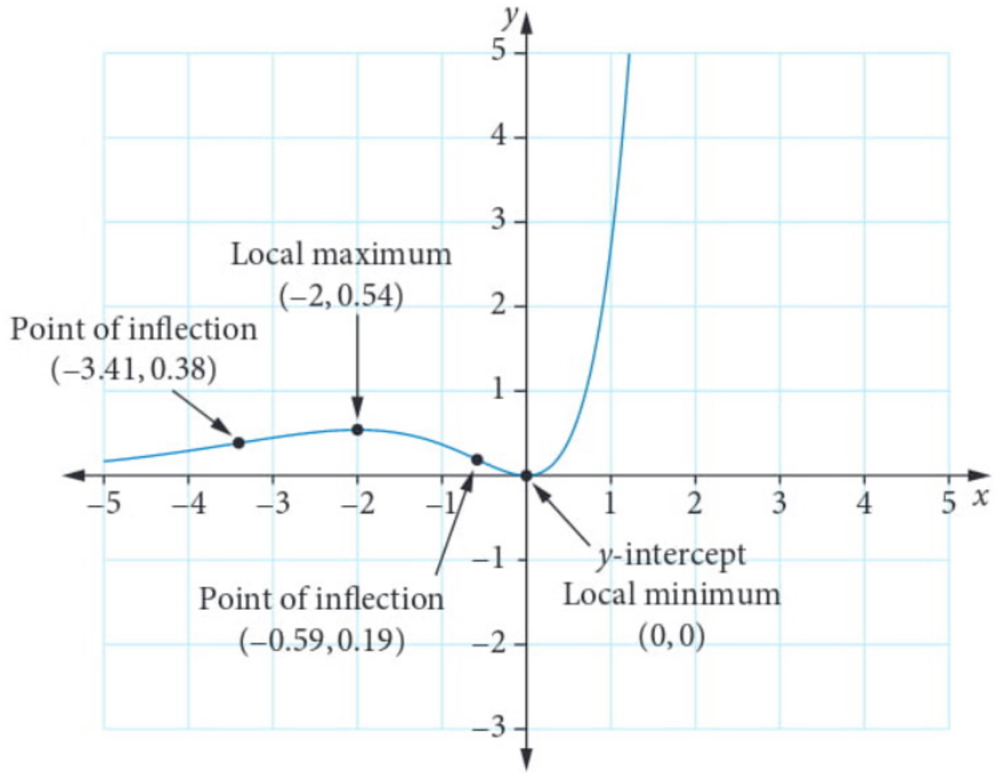
c **points of inflection $\boldsymbol{f''(x) = 0}$** ✓

$e^x(x^2 + 4x + 2) = 0$

$$x = \frac{-4 \pm \sqrt{16-8}}{2} = -2 \pm \sqrt{2}$$

Points are $(\mathbf{-3.4, 0.38})$ and $(\mathbf{-0.59, 0.19})$. ✓

d



indicates local minimum ✓

indicates local maximum ✓

indicates points of inflections ✓

overall shape ✓

EXERCISE 4.5 Integrals of trigonometric functions

ANSWERS p. 395

Recap

1 An object's velocity v m/s at time t s is $v = \frac{3}{4}\tan(8t - \pi)$. Find the object's acceleration in m/s^2 when $t = \frac{5\pi}{32}$.

2 If $\frac{dy}{dx} = \sqrt{2}\cos(\pi - 3x)$, find the value of x when y is a maximum for $0 < x < \pi$.

Mastery

3 WORKED EXAMPLE 22 Find

a $\int 6\sin(2x)\,dx$

b $\int \frac{1}{2}\cos\left(\frac{x}{2}\right)dx$

c $\int 3\sin\left(\frac{1}{2}(5x - 7)\right)dx$

d $\int_0^{\frac{\pi}{12}} 2\cos(2x)\,dx$

e $\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} 4\sin(2x)\,dx$

f $\int_0^{\frac{\pi}{12}} \sin(2x) - \cos(3x)\,dx$

4 WORKED EXAMPLE 23 Given that $\frac{d}{dx}(3x\sin(2x)) = 3\sin(2x) + 6x\cos(2x)$, write the expression for $\int 12x\cos(2x)\,dx$.

5 WORKED EXAMPLE 24 If $f'(x) = \sin(3x) + \cos(x)$ and $f\left(\frac{\pi}{2}\right) = 3$, find the function $f(x)$.

6 WORKED EXAMPLE 25 Find the area bounded by the graph $y = 4\sin(3x)$ between $x = 0$ and $x = \pi$.

7 WORKED EXAMPLE 26 It takes an elevator 40 seconds to ascend from the ground floor of a building to the tenth floor. The velocity of the elevator during its ascent is given by

$$v(t) = \frac{7\pi}{8}\sin\left(\frac{\pi t}{40}\right)\text{m/s}.$$

The velocity, v, is measured in metres per second and the time, t, is measured in seconds.

Find

a the acceleration of the elevator in terms of t

b $\int_0^{\frac{20}{3}} v'(t)\,dt$

c the displacement function of the elevator if the ground floor has displacement zero metres.

8 Using CAS 6

Find, correct to two decimal places, the area of the two regions bounded by the curves $y = \sin(2x)$ and $y = \cos(x)$ in the interval $0 \le x \le \frac{\pi}{2}$.

Calculator-free

9 (4 marks)

a Find an anti-derivative of $\cos(2x + 1)$ with respect to x. (1 mark)

b If $f'(x) = 2\cos(x) - \sin(2x)$ and $f\left(\frac{\pi}{2}\right) = \frac{1}{2}$, find $f(x)$. (3 marks)

10 (3 marks) If $f(x) = x\cos(3x)$, then $f'(x) = \cos(3x) - 3x\sin(3x)$.

Use this fact to find the anti-derivative of $x\sin(3x)$.

11 (2 marks) The acceleration a m/s^2 of a particle at time t s is given by $a(t) = 2\sin(t) - 18\cos(3t)$. If the speed of the particle is 0 m/s after π seconds and its position is $-\pi$ m after $\frac{\pi}{2}$ s, determine the equation for the object's position.

12 (2 marks) The motion of an object is described by $a(t) = \cos(t)$, where $a(t)$ is the object's acceleration at time t. Determine the equation for the object's position, $s(t)$, given that $s\left(\frac{\pi}{2}\right) = \frac{\pi}{2} + 2$ and $s(\pi) = \pi + 3$.

13 © SCSA MM2017 Q8 (5 marks)

a Differentiate $2x\sin(3x)$ with respect to x. (2 marks)

b Hence show that $\int x\cos(3x)\,dx = \frac{3x\sin(3x) + \cos(3x)}{9} + c$. (3 marks)

14 © SCSA MM2019 Q9 (8 marks) It takes an elevator 16 seconds to ascend from the ground floor of a building to the sixth floor. The velocity of the elevator during its ascent is given by

$$v(t) = \frac{9\pi}{16}\sin\left(\frac{\pi t}{16}\right)\text{m/s.}$$

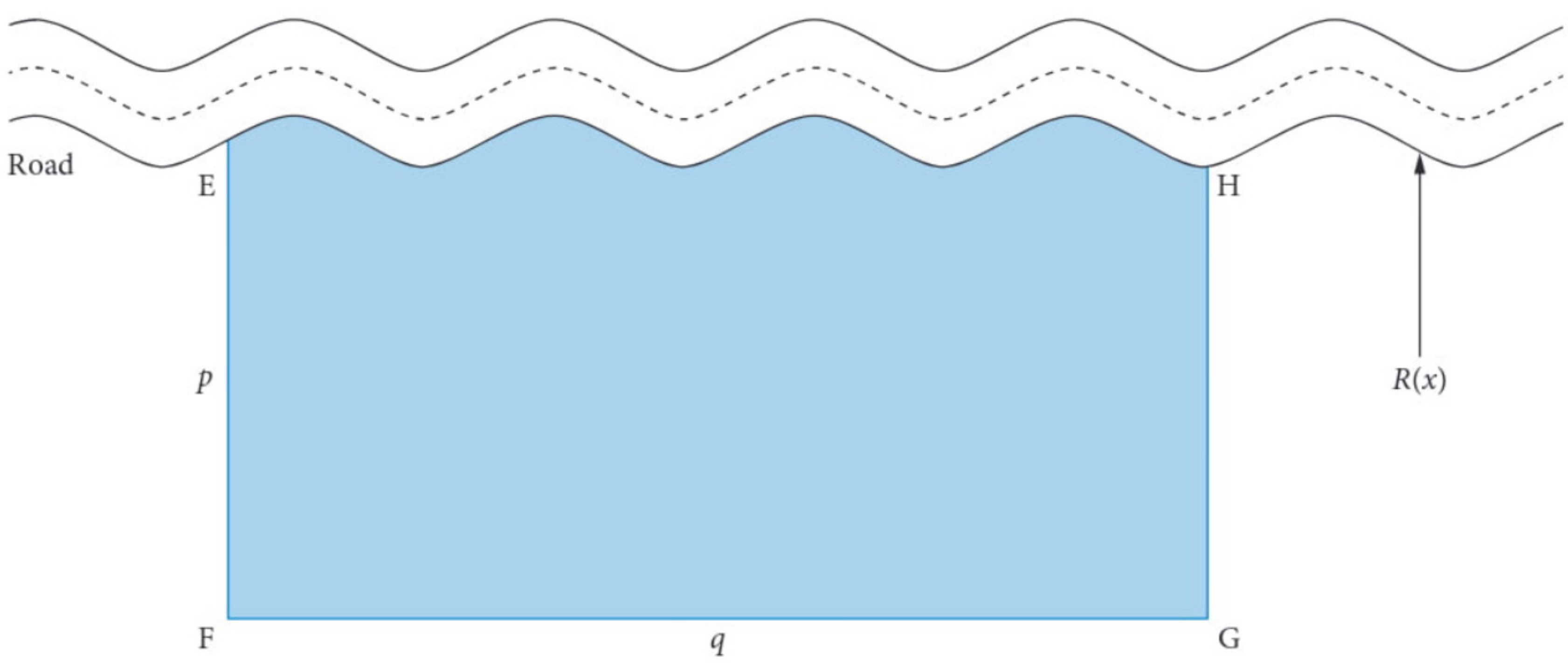
The velocity, v, is measured in metres per second, while the time, t, is measured in seconds.

a Determine the acceleration of the elevator during its ascent and provide a sketch of the acceleration function for $0 \leq t \leq 16$. (2 marks)

b With reference to your answer from part **a**, explain what is happening to the velocity of the elevator in the interval $0 < t < 8$ and in the interval $8 < t < 16$. (3 marks)

c Suppose that the ground floor has displacement $x = 0$ m. Determine the displacement function of the elevator and hence determine the height above the ground floor of the sixth floor. (3 marks)

15 © SCSA MM2020 Q17ab (8 marks) David and Katrina have a small farm and wish to fence off an area of their land so they can raise sheep. The area they have chosen has one border along a road as shown in the diagram below.



The enclosure is shown as the shaded area above and has right angles at points F and G. David and Katrina want the combined lengths of the fencing from E to F and F to G to equal 500 metres. Let the length of fence EF be equal to p metres and the length of fence FG be equal to q metres. If we locate the origin at point F and the x-axis along the line FG, the equation defining the fence along the road is given by:

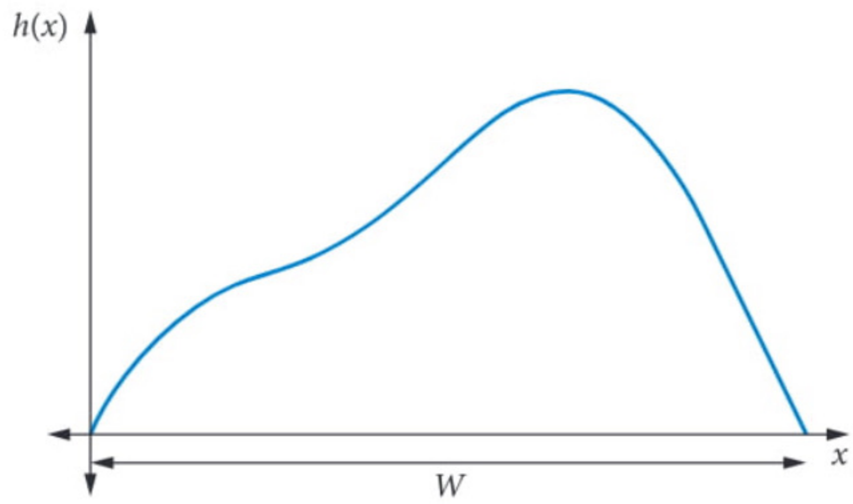
$$R(x) = 10\sin\left(\frac{x}{15}\right) + p$$

a Show that the equation defining the area of the enclosure, $A(q)$, can be given in terms of q as follows:

$$A(q) = 500q - 150\cos\left(\frac{q}{15}\right) - q^2 + 150$$ (4 marks)

b Determine, to the nearest metre, the value of q that will allow the sheep to graze over the maximum area and state this maximum area. (4 marks)

16 © SCSA | MM2021 Q9abc (5 marks) The Interesting Architecture company has designed a building with a constant cross-section shown in the figure below.



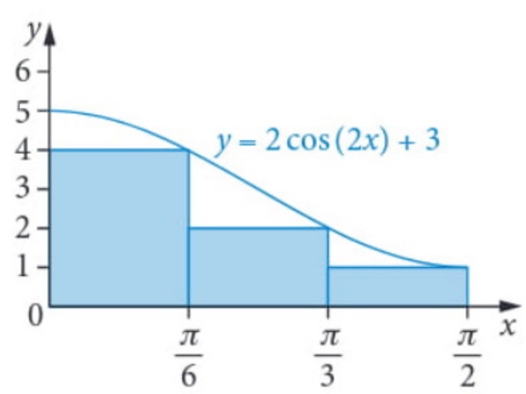
With reference to the figure, the height $h(x)$ of the building at a point x along its width is given by $h(x) = 4\sin\left(x - \frac{3\pi}{2}\right) - x^2 + 3\pi x - 4$ where h and $0 \le x \le W$ are measured in metres.

a Determine the width W of the building to the nearest centimetre. (2 marks)

b Determine $h'(x)$. (1 mark)

c Determine, to the nearest centimetre, the value of x at which the height of the building is maximum and state this maximum height. (2 marks)

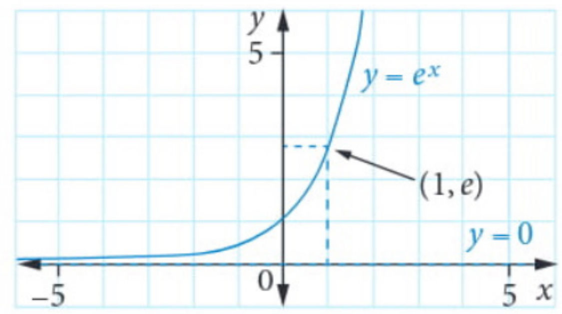
17 (2 marks) Jamie approximates the area between the x-axis and the graph of $y = 2\cos(2x) + 3$, over the interval $0 \le x \le \frac{\pi}{2}$, using the three rectangles shown below.



Determine Jamie's approximation as a fraction of the exact area.

4 Chapter summary

The natural exponential function $y = e^x$



- $e = 2.718\,28\ldots$ is Euler's number.
- Euler's number is defined as $\lim\limits_{n \to \infty}\left(1+\frac{1}{n}\right)^n$. As $\lim\limits_{n \to \infty}\left(1+\frac{x}{n}\right)^n = e^x$.
- The gradient of the graph is always increasing.
- The y-intercept is 1 (because $e^0 = 1$).
- The x-axis ($y = 0$) is a horizontal asymptote.

Exponential growth and decay

- The general form of an **exponential growth or decay function** is $N(t) = N_0e^{kt}$, where N_0 is the initial value and the value of k determines the rate of growth or decay.
- For a **growth function**, $k > 0$.
- For a **decay function**, $k < 0$.

Differentiating exponential functions

- $\frac{d}{dx}(e^x) = e^x$
- $\frac{d}{dx}(e^{ax-b}) = ae^{ax-b}$
- If $y = e^{f(x)}$, then $\frac{dy}{dx} = f'(x)e^{f(x)}$ by the chain rule.

Integrating exponential functions

- $\int e^x\,dx = e^x + c$
- $\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c$

Derivatives of trigonometric functions

Trigonometric function	Derivative
$y = \sin(ax)$	$\frac{dy}{dx} = a\cos(ax)$
$y = \cos(ax)$	$\frac{dy}{dx} = -a\sin(ax)$

Integrals of trigonometric functions

$$\int \sin(ax-b)\,dx = -\frac{1}{a}\cos(ax-b) + c$$

$$\int \cos(ax-b)\,dx = \frac{1}{a}\sin(ax-b) + c$$

Straight line motion

- displacement x at time t: $x(t) = \int v(t)\,dt$
- velocity v at time t: $v(t) = \frac{dx}{dt}$ or $v(t) = \int a(t)\,dt$
- acceleration a at time t: $a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}$

Cumulative examination: Calculator-free

Total number of marks: 20 Reading time: 2 minutes Working time: 20 minutes

1 (6 marks)

Consider the function $y = (x + 1)(x^2 - 3x + 3)$.

a Determine the gradient of the tangent to the curve at $x = 3$. (3 marks)

b Using calculus techniques, determine the nature of the stationary point at $x = \frac{4}{3}$. (3 marks)

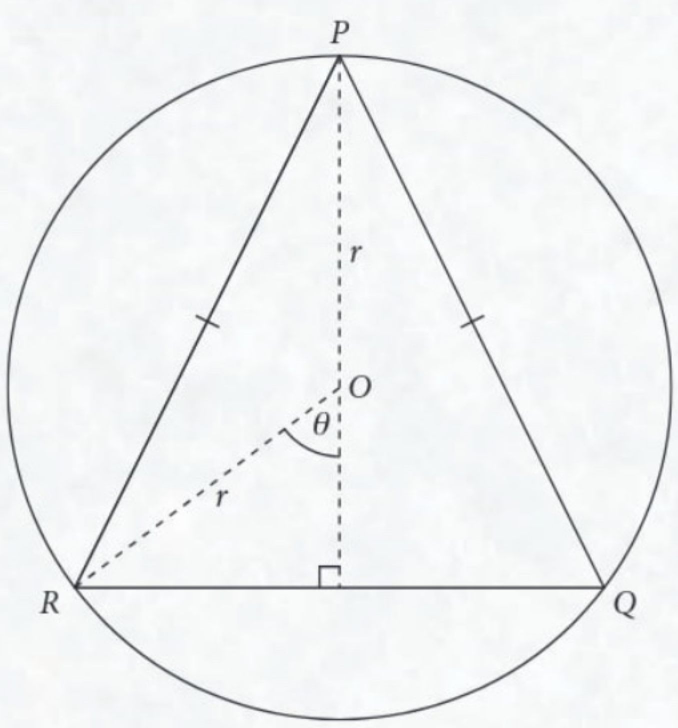
2 © SCSA MM2018 Q3a–ci (7 marks)

a Differentiate $(2x^3 + 1)^5$. (2 marks)

b Given $g'(x) = e^{2x}\sin(3x)$, determine a simplified value for the rate of change of $g'(x)$ when $x = \frac{\pi}{2}$. (3 marks)

c Determine $\int 3\cos(2x)\,dx$. (2 marks)

3 © SCSA MM2016 Q8 (7 marks) An isosceles triangle ΔPQR is inscribed inside a circle of fixed radius r and centre O. Let θ be defined as in the diagram below.



a Show that the area A of the triangle ΔPQR is given by $A = r^2\sin\theta(1 + \cos\theta)$. (2 marks)

b Using calculus, determine the value of θ that maximises the area A of the inscribed triangle. State this area in terms of r exactly. Justify your answer. (Hint: you may need the identity $\sin^2 x = 1 - \cos^2 x$ in your working.) (5 marks)

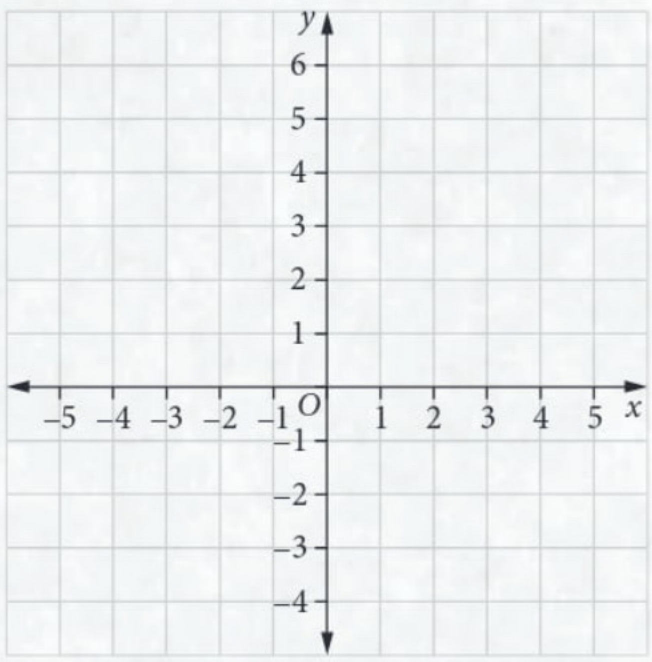
Cumulative examination: Calculator-assumed

Total number of marks: 21 Reading time: 3 minutes Working time: 21 minutes

1 (5 marks) Let $f(x) = \frac{2}{(x-1)^2} + 1$.

a **i** Evaluate $f(-1)$. (1 mark)

ii Copy the axes below and sketch the graph of f, labelling all asymptotes with their equations. (2 marks)



b Find the area bounded by the graph of f, the x-axis, the line $x = -1$ and the line $x = 0$. (2 marks)

2 © SCSA MM2017 Q16 (8 marks) A group of biologists has decided that colonies of a native Australian animal are in danger if their populations are less than 1000. One such colony had a population of 2300 at the start of 2011. The population was growing continuously such that $P = P_0e^{0.065t}$ where P is the number of animals in the colony t years after the start of 2011.

a Determine, to the nearest 10 animals, the population of the colony at the start of 2014. (2 marks)

b Determine the rate of change of the colony's population when $t = 2.5$ years. (2 marks)

c At the beginning of 2017, a disease caused the colony's population to decrease continuously at the rate of 8.25% of the population per year. If this rate continues, when will the colony become 'in danger'? Give your answer to the nearest month. (4 marks)

3 © SCSA MM2018 Q11 (8 marks) Ava is flying a drone in a large open space at a constant height of 5 metres above the ground. She flies the drone due north so that it passes directly over her head and then, sometime later, reverses its direction and flies the drone due south so it passes directly over her again. With $t = 0$ defined as the moment when the drone first flies directly above Ava's head, the velocity of the drone, at time t seconds, is given by:

$$v = 2\sin\left(\frac{t}{3} + \frac{\pi}{6}\right)\text{m/s} \quad 0 \le t \le 16$$

a Determine $x(t)$, the displacement of the drone at t seconds, where $x(0) = 0$. (3 marks)

b Where is the drone in relation to the pilot after 16 seconds? (2 marks)

c At a particular time, the drone is heading due south and it is decelerating at 0.5 m/s^2. How far has the drone travelled from its initial position directly above Ava's head until this particular time? (3 marks)